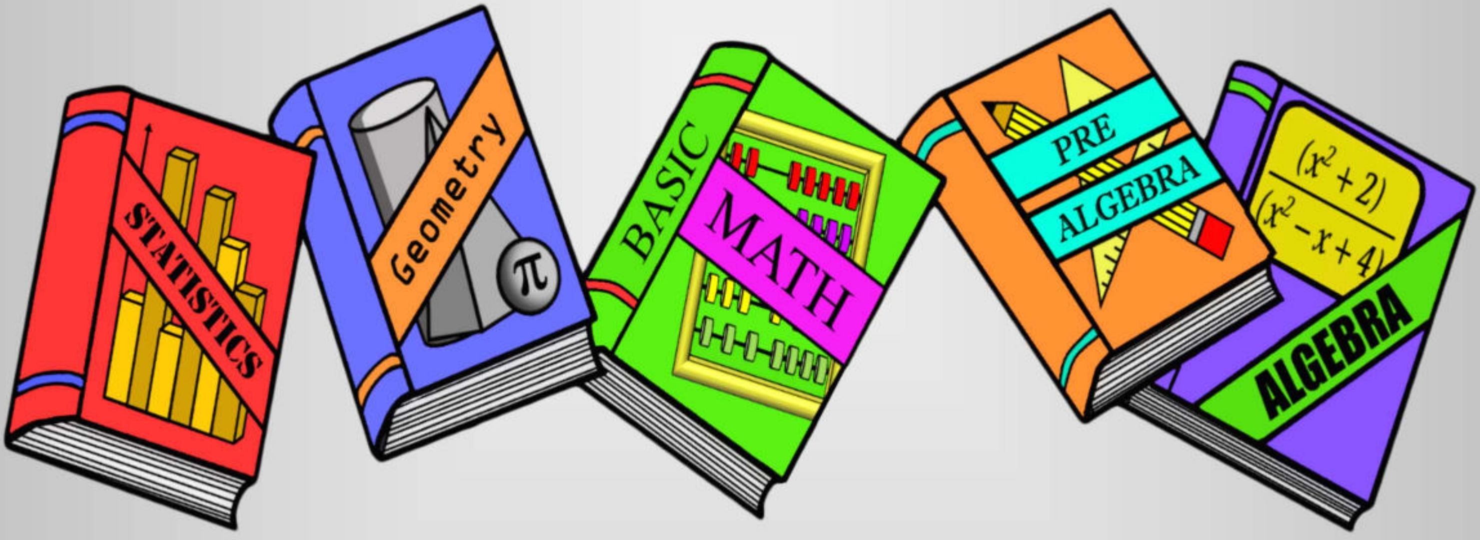
PROVEN AND TIME-TESTED METHODS!

MA+h on the Fly!

ACT® MATH

2024–2025 PREP BOOK

COMPLETE, SIMPLE AND EFFECTIVE, LIKE IT SHOULD BE!



LESSONS THAT BUILD CONFIDENCE!

Explains the Basics and Advanced Topics!

50+ Sections with Step-by-Step Examples

150+ Printed ACT Concepts for Quick Review

1,100+ Problems with Complete Solutions

NATHANIEL BROWN

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About the Author and Book:

Nathaniel Brown is a Computer Engineering graduate from The University of Alabama.
He is an educator and has assisted students on the Math portion of the ACT for nearly 15 years.

Front page logos were designed by James Lowrey and Linda Holman.
Book layout, lessons and content were designed by Nathaniel Brown.

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Introduction and General Test Strategy

INTRODUCTION:

Hello and welcome! Many students freak out over the math section of the ACT. We can assure you that the test is nowhere near as scary as people make it out to be. In fact, if you answer about 50% of the math questions correctly, you would have scored better than half of all ACT test takers in the entire United States (ACT National Profile Report, 2018). You do not have to be spectacular to get a good ACT score. At the end of the day, it is just a test. It doesn't define your life or your value as person, so there's no need to get overly stressed out about it. Study what you can from the book, soak in the information, and just do your best.

This math book is based around one central idea: A thorough and strong math foundation leads to better test scores. As you go through the book, you will see that we have taken time to break down every concept in detail. We provide hundreds of examples, over 1,000 practice problems, as well as notecards and detailed solutions to help you. Over 2,000 actual ACT test questions over the past 10 years were analyzed to help create the lessons and problems that are presented in this book. We will cover tricks and strategy and blend it with the basics and fundamentals to help you think like the test makers and students who score high on the test.

As an educator, our job is to serve YOU. If you are willing to work and practice the concepts displayed in this book, it is guaranteed you will see better test scores and results. Anybody, regardless of age, can get better in mathematics.

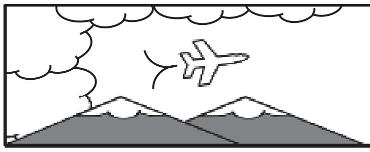
GENERAL TEST STRATEGY:

For the math section on the test, you are given 60 minutes to answer 60 multiple–choice questions. That is not a lot of time. Since the questions vary in difficulty, do not spend too much time on one problem. Always answer questions you can easily solve first. Then go back and handle the harder questions. This is a good strategy for any test you take, regardless of math.

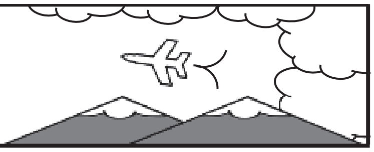
On the ACT, you do not get penalized for wrong answers. Therefore, you need to answer every question on the test, including the questions you have no idea how to solve. Make sure you give yourself enough time at the end of the test to answer any questions you may have skipped over. Look for opportunities to eliminate answer choices to improve your chances of getting questions correct.

In addition, the ACT *does not* provide students a formula sheet for the test. You will need to memorize most of the math formulas that are used on the test, and we offer our own formula sheets to help you.

Outside of this book, you need to take as many ACT practice tests as you can. Many ACT practice tests can be found online. If you have not bought The Official ACT Prep Guide by the makers of the ACT, it is recommended you get a copy. If the print version is too expensive, the cheaper eBook version is just as good. Get comfortable with the wording and patterns of how math questions are presented on the test.



Calculator Quiz



SCIENTIFIC CALCULATORS:

Even though you do not need a calculator to take the ACT, we highly recommend you bring one to the test. Scientific calculators are typically the best calculators to use since they are more powerful than a basic four function calculator and easier to use than expensive graphing calculators. (If you are a student that is currently using a graphing calculator, that is perfectly fine, as long as the calculator meets ACT calculator guidelines. Just make sure that you are confident in using the calculator by the time you take the test.)

A good scientific calculator is the equivalent of having a math tutor or math teacher right next to you helping you work out all the calculations. Once you learn how to use one good scientific calculator, you will pretty much know how to use any scientific calculator on the market. Calculator manuals, Google, YouTube and other internet sources provide excellent tutorials and tips on how to use your specific brand of calculator. Don't be the person that tries to learn their calculator 10 minutes before the test starts. You know better than that.

Good scientific calculators have the following characteristics:
1. They are affordable ($25 or less) and are widely available for purchase in general retail and online outlets.
2. They allow you to type a math problem on the calculator screen exactly how you see it on paper.
3. They can work out all operations involving fractions and mixed numbers.
4. They can convert between fractions and decimals.
5. They can simplify numbers raised to exponents, including negative powers and fractional exponents.
6. They can simplify expressions involving square roots.
7. They allow for quick computations of logarithms.
8. Trigonometric functions can be easily calculated with angles expressed in degrees or radians.

To help you get comfortable with your calculator, take the quiz below. Work out every single problem using ONLY your calculator. If there are problems below that your calculator can't perform, it is 100% ok. That just means you will need to learn how to work out those problems by hand, which we will cover in the book. If it takes you a long time to complete the quiz, retake it until you can complete the quiz quickly and efficiently. You cannot afford to waste time with a calculator on a 60-minute test. Be an expert of your calculator.

CALCULATOR QUIZ:

1. Simplify: $2-3(-6+4)^2$
2. Change 0.625 to a reduced fraction.
3. Simplify: 3^{12}
4. Simplify: $81^{5/4}$
5. Write the answer as a fraction: 9^{-3}
6. Write the answer as a fraction: $7/9+8/11-1/2$
7. Write the answer as a fraction: $8\,\frac{5}{6}\div 2\,\frac{4}{9}$
8. Simplify: $\sqrt{676}-\sqrt{1600}$
9. Simplify: $\sqrt[3]{512}+\sqrt[4]{256}$
10. Put $\sqrt{72}$ in simplest radical form.
11. Simplify: $\log 1000$
12. Simplify: $\log_2 128$. If you can't plug it into your calculator, just use $\log(128)\div\log(2)$.

13. Follow steps A through C:
A. Make sure your calculator is in "Degree" mode. Plug in $\sin(30)$ and get the answer.
B. Now change your calculator to "Radians" mode. Plug in $\sin(\pi/6)$ and get the answer.
C. Change your calculator back to "Degree" mode. Plug in $\tan(135)$ and get the answer.

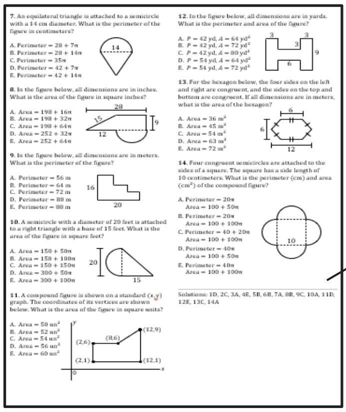
Solutions to Calculator Quiz:
1. –10, **2.** 5/8, **3.** 531441, **4.** 243, **5.** 1/729, **6.** 199/198, **7.** 159/44, **8.** –14, **9.** 12, **10.** $6\sqrt{2}$, **11.** 3, **12.** 7, **13A.** 1/2 or 0.5, **13B.** 1/2 or 0.5 **13C.** –1

Important Features of the Book

As you make your way through the sections of the book, you will notice several unique features:

Practice Problems and Extended Solutions:

Practice Problems:



7. An equilateral triangle is attached to a semicircle with a 14 cm diameter. What is the perimeter of the figure in centimeters?

A. Perimeter = 28 + 7π
B. Perimeter = 28 + 14π
C. Perimeter = 35π
D. Perimeter = 42 + 7π
E. Perimeter = 42 + 14π

8. In the figure below, all dimensions are in inches. What is the area of the figure in square inches?

A. Area = 198 + 16π
B. Area = 198 + 32π
C. Area = 198 + 64π
D. Area = 252 + 32π
E. Area = 252 + 64π

9. In the figure below, all dimensions are in meters. What is the perimeter of the figure?

A. Perimeter = 56 m
B. Perimeter = 64 m
C. Perimeter = 72 m
D. Perimeter = 80 m
E. Perimeter = 88 m

10. A semicircle with a diameter of 20 feet is attached to a right triangle with a base of 15 feet. What is the area of the figure in square feet?

A. Area = 150 + 50π
B. Area = 150 + 100π
C. Area = 150 + 150π
D. Area = 300 + 50π
E. Area = 300 + 100π

11. A compound figure is shown on a standard (x,y) graph. The coordinates of its vertices are shown below. What is the area of the figure in square units?

A. Area = 50 un²
B. Area = 52 un²
C. Area = 54 un²
D. Area = 56 un²
E. Area = 60 un²

12. In the figure below, all dimensions are in yards. What is the perimeter and area of the figure?

A. $P = 42$ yd, $A = 64$ yd²
B. $P = 42$ yd, $A = 72$ yd²
C. $P = 42$ yd, $A = 80$ yd²
D. $P = 54$ yd, $A = 64$ yd²
E. $P = 54$ yd, $A = 72$ yd²

13. For the hexagon below, the four sides on the left and right are congruent, and the sides on the top and bottom are congruent. If all dimensions are in meters, what is the area of the hexagon?

A. Area = 36 m²
B. Area = 45 m²
C. Area = 54 m²
D. Area = 63 m²
E. Area = 72 m²

14. Four congruent semicircles are attached to the sides of a square. The square has a side length of 10 centimeters. What is the perimeter (cm) and area (cm²) of the compound figure?

A. Perimeter = 20π
Area = 100 + 50π
B. Perimeter = 20π
Area = 100 + 100π
C. Perimeter = 40 + 20π
Area = 100 + 100π
D. Perimeter = 40π
Area = 100 + 50π
E. Perimeter = 40π
Area = 100 + 100π

Solutions: 1D, 2C, 3A, 4E, 5B, 6B, 7A, 8B, 9C, 10A, 11D, 12E, 13C, 14A

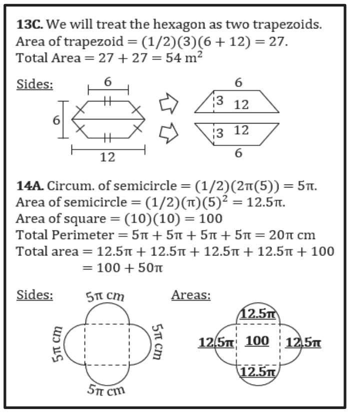
At the end of each section are practice problems.
The solutions are located at the end of the section.

Practice Problems

Solutions: 1D, 2C, 3A, 4E, 5B, 6B, 7A, 8B, 9C, 10A, 11D, 12E, 13C, 14A

If you are unsure how an answer was derived,
you can go to the back of the book to get
a detailed explanation of the solution.

Extended Solutions:



13C. We will treat the hexagon as two trapezoids.
Area of trapezoid = (1/2)(3)(6 + 12) = 27.
Total Area = 27 + 27 = 54 m²

Sides:

14A. Circum. of semicircle = (1/2)(2π(5)) = 5π.
Area of semicircle = (1/2)(π)(5)² = 12.5π.
Area of square = (10)(10) = 100
Total Perimeter = 5π + 5π + 5π + 5π = 20π cm
Total area = 12.5π + 12.5π + 12.5π + 12.5π + 100
= 100 + 50π

Sides: Areas:

30+ Sections:

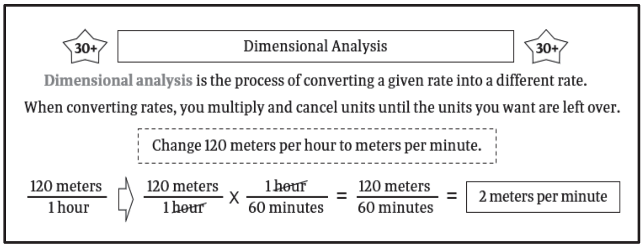
30+ In the book, you will see stars with “30+” on them. 30+
A small number of questions on the ACT come from those headings.

These parts of the book should be studied if you are aiming for a test score
of 30 or higher on the ACT math test.

You can still get a good score without knowing the material under 30+ headings.
Just keep in mind that it is those questions that separate students who get
good scores versus students who achieve exceptionally high scores on the test.

If you see a 30+ star next to the title at the very top of a section, then *everything* in the section is 30+.

Most of these types of sections are found in Chapter 5 and Chapter 6.

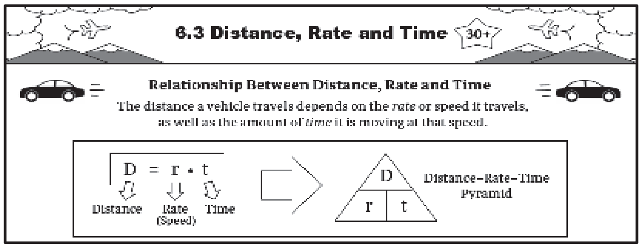


30+ Dimensional Analysis 30+

Dimensional analysis is the process of converting a given rate into a different rate.
When converting rates, you multiply and cancel units until the units you want are left over.

Change 120 meters per hour to meters per minute.

$$\frac{120\text{ meters}}{1\text{ hour}} \Rightarrow \frac{120\text{ meters}}{1\cancel{\text{ hour}}} \times \frac{1\cancel{\text{ hour}}}{60\text{ minutes}} = \frac{120\text{ meters}}{60\text{ minutes}} = \boxed{2\text{ meters per minute}}$$



6.3 Distance, Rate and Time 30+

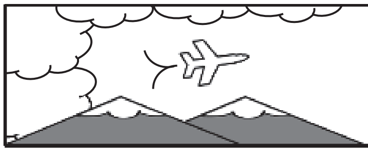
Relationship Between Distance, Rate and Time

The distance a vehicle travels depends on the *rate* or speed it travels,
as well as the amount of *time* it is moving at that speed.

We recommend all students take at least a small look at some of the 30+ headings and flash cards.

Some of the 30+ concepts are simple to learn such as solving basic logarithms,
adding and subtracting matrices, and so on.

We hope you benefit greatly from the book. Enjoy!



Common ACT Math Formulas

Area and Perimeter:

Rectangle or Square: $A = LW$ or $A = bh$

Square: $A = s^2$

Parallelogram: $A = bh$

Triangle: $A = \frac{1}{2}bh$

Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$

Circle (Circumference): $C = 2\pi r$ or $C = \pi d$

Circle (Area): $A = \pi r^2$

Volume:

Rectangular Prism or Cube: $V = LWH$

Cube: $V = s^3$

Prism: $V = Bh$

Cylinder: $V = \pi r^2 h$

Coordinate Geometry:

Distance Formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint Formula: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Slope Formula: $m = \frac{\text{Rise}}{\text{Run}} = \frac{y_1 - y_2}{x_1 - x_2}$

$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$

Slope–Intercept Form: $y = mx + b$

Pythagorean Theorem:

$a^2 + b^2 = c^2$

SOH–CAH–TOA:

$\cos(x) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

$\sin(x) = \frac{\text{Opposite}}{\text{Hypotenuse}}$

$\tan(x) = \frac{\text{Opposite}}{\text{Adjacent}}$

Equation of a Circle:

$(x - h)^2 + (y - k)^2 = r^2$
where (h,k) = center, r = radius

Rules of Exponents:

$x^A \bullet x^B = x^{A+B}$

$\frac{x^A}{x^B} = x^{A-B}$

$(x^A)^B = x^{A \bullet B}$

$(xyz)^A = x^A y^A z^A$

$x^0 = 1$

$x^{-A} = \frac{1}{x^A}$

$\left(\frac{X}{Y}\right)^A = \frac{X^A}{Y^A}$

$\left(\frac{X}{Y}\right)^{-A} = \left(\frac{Y}{X}\right)^A$

Rules of Radicals:

$\sqrt[n]{A \bullet B} = \sqrt[n]{A} \bullet \sqrt[n]{B}$

$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$

Percent Change:

$\frac{\text{New Value} - \text{Old Value}}{\text{Old Value}} \times 100$

Difference of Squares:

$A^2 - B^2 = (A - B)(A + B)$

Sum of Interior Angles of Polygons

Triangle: Angles add up to 180^o

Quadrilateral: Angles add up to 360^o

Pentagon: Angles add up to 540^o

Any Polygon: $S = 180(n - 2)$ where n = # of sides

Probability:

$\text{Probability} = \frac{\text{Items You Need}}{\text{Total Possibilities}}$

Statistics:

$\text{Mean} = \frac{\text{Sum of the Numbers}}{\text{Number of Values}}$

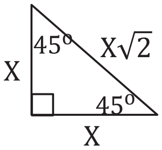
Median = The number in the middle of a list
(If two numbers are in the middle, add and divide by 2)

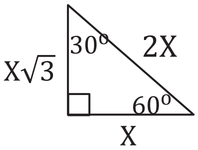
Mode = Number that occurs the most

Range = Biggest number − Smallest number

Other ACT Math Formulas 30+

Special Right Triangles:





Surface Area:

Rectangular Prism: $SA = 2LW + 2WH + 2LH$

Cube: $SA = 6s^2$

Classifying Triangles:

Acute Triangle: $a^2 + b^2 > c^2$

Right Triangle: $a^2 + b^2 = c^2$

Obtuse Triangle: $a^2 + b^2 < c^2$

Ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Arc Length: $L = \frac{X^o}{360}(2\pi r)$

Trigonometry Formulas:

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

Area of a Triangle: $A = \frac{1}{2}ab\sin C$

Trigonometric Identities:

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$\sec(x) = \frac{1}{\cos(x)}$, $\csc(x) = \frac{1}{\sin(x)}$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\cot(x) = \frac{\cos(x)}{\sin(x)}$

Complex and Imaginary Numbers:

$i^1 = \sqrt{-1} = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Sequences and Series:

Arithmetic Sequence: $a_n = a_1 + d(n-1)$

Sum of Arithmetic Series: $s = \frac{n(a_1 + a_L)}{2}$

Sum and Difference of Cubes:

$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

Quadratic Formula and Discriminant:

(Assumes the quadratic equation is $Ax^2 + Bx + C = 0$)

Quadratic Formula: $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

Discriminant: $D = B^2 - 4AC$

{ $D > 0$ (2 Real Sol.), $D = 0$ (1 Real Sol.), $D < 0$ (0 Real Sol.) }

Rules of Radicals (Fractional Exponents):

$\sqrt[n]{X} = X^{1/n}$

$X^{A/B} = \sqrt[B]{X^A}$ or $(\sqrt[B]{X})^A$

Direct and Inverse Variation:

Direct Variation: $y = kx$

Inverse Variation: $y = \frac{k}{x}$

Counting and Arrangements:

Combinations: $C(n,r) = \frac{n!}{r!\,(n-r)!}$

Permutations: $P(n,r) = \frac{n!}{(n-r)!}$

Distinguishable Permutations: $\frac{n!}{n_1!\,n_2!\,n_3!\ldots}$

Other Rules of Probability:

Addition Rule of Probability:
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Mutually Exclusive Events:
$P(A \text{ or } B) = P(A) + P(B)$
$P(A \text{ and } B) = 0$

Independent Events:
$P(A \text{ and } B) = P(A) \bullet P(B)$

Expected Value: $\sum x \bullet p(x)$

Distance, Rate, Time: $d = rt$

Logarithms:

$\log_a b = x \rightarrow a^x = b$

$\log_a(xy) = \log_a(x) + \log_a(y)$

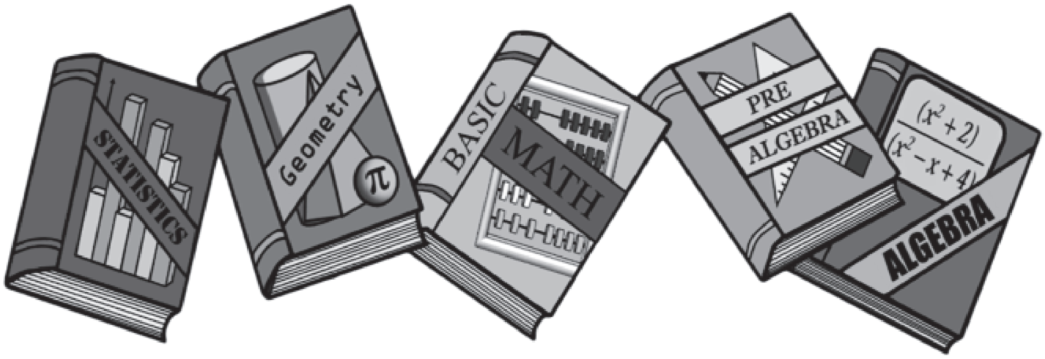
$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

$\log_a(x^c) = c\log_a(x)$

Math on the Fly!

CHAPTER 1

Pre–Algebra and Fundamentals



1.1 Sets and Groups of Numbers

We start off by showing the most common groups of numbers that show up on the ACT.

Complex Numbers

Complex numbers include ALL real numbers and imaginary numbers.

Imaginary Numbers

We can't take the square root
of a negative number.

We use the letter 'i' when a number is imaginary.

$\sqrt{49} = 7$ (The 7 is real.)

$\sqrt{-49} = 7i$ (The 7 is imaginary.)

Real Numbers

If a number has no imaginary parts, it is real.

$\sqrt{49} = 7$ $\sqrt{3}$ π

-6 6 6.1 $6.1111\ldots = 6.1\overline{1}$ $\frac{1}{2}$ $\frac{8}{11}$

Real Numbers

Real numbers include all rational and irrational numbers.

Irrational Numbers

These numbers cannot be written as a fraction.

As decimals, they go on forever
but never in a pattern.

Imperfect square roots and Pi are irrational:

$\sqrt{22} \approx 4.6904158\ldots$ $\pi \approx 3.1415927\ldots$

Rational Numbers

If a number is not irrational, it is rational.

These numbers can be written as a fraction.
As decimals, they either stop or repeat forever.

$\frac{27}{4}$ 6.75 $\frac{7}{12} = 0.583333\ldots = 0.58\overline{3}$ $\frac{-9}{1} = -9$

Integers

Integers are all the positive and negative numbers
you normally put on a number line.
…–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5…

Natural Numbers

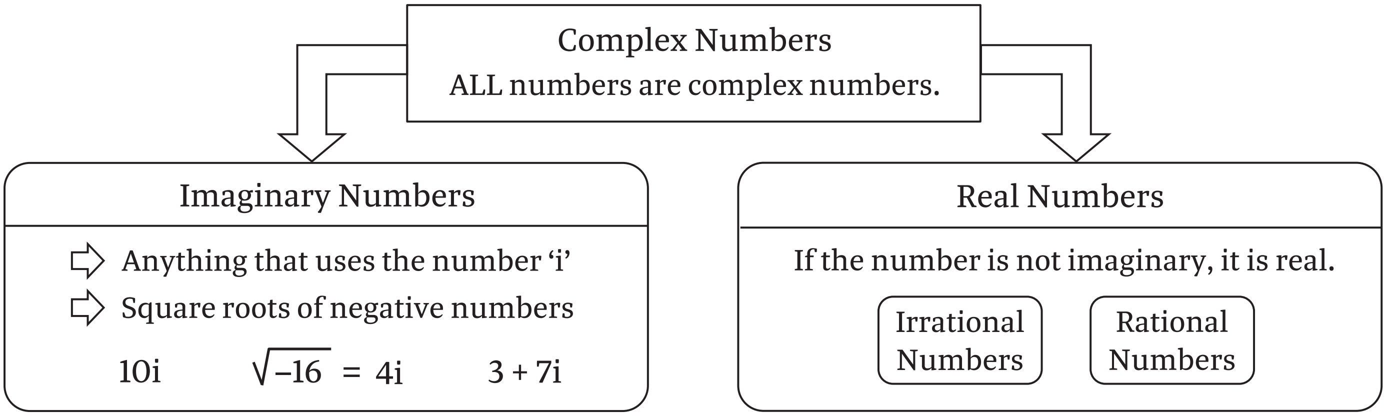
Natural numbers are the positive numbers
that are used for counting.
1, 2, 3, 4, 5, 6, 7, 8, 9, 10…

Whole Numbers

Whole numbers are all the natural numbers,
and zero.
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10…

Overview of Sets of Numbers

The charts below are here to help you get a better idea of what groups a number belongs to.



Complex Numbers

ALL numbers are complex numbers.

Imaginary Numbers

- Anything that uses the number 'i'
- Square roots of negative numbers

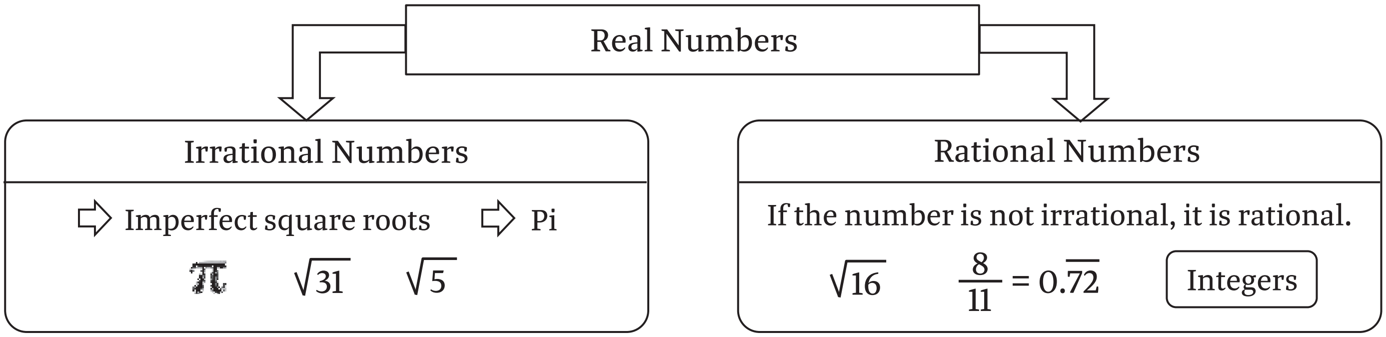
10i $\sqrt{-16} = 4i$ 3 + 7i

Real Numbers

If the number is not imaginary, it is real.

Irrational Numbers

Rational Numbers



Real Numbers

Irrational Numbers

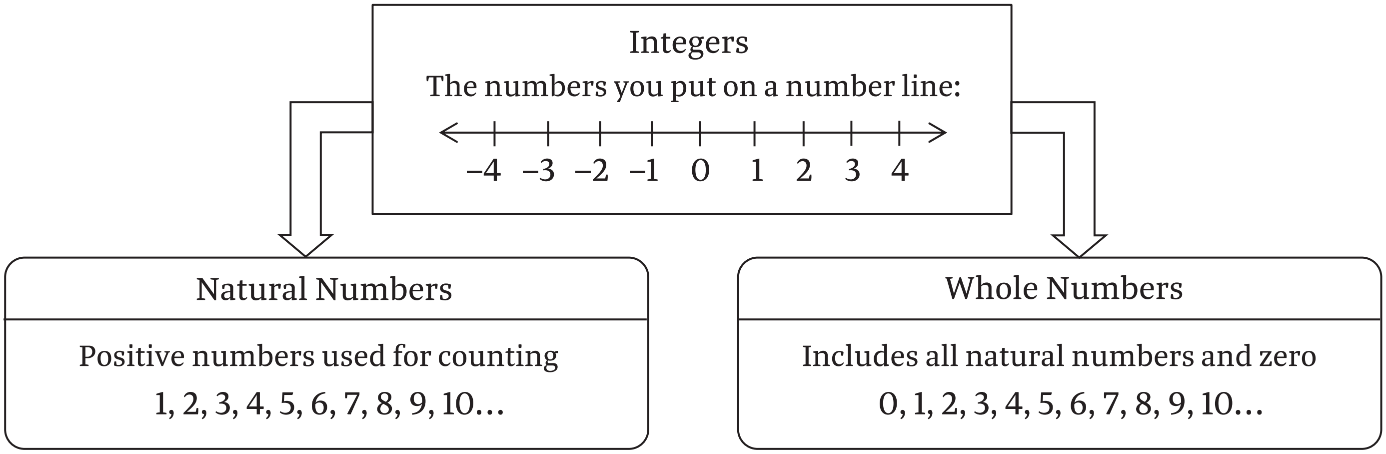
- Imperfect square roots
- Pi

π $\sqrt{31}$ $\sqrt{5}$

Rational Numbers

If the number is not irrational, it is rational.

$\sqrt{16}$ $\frac{8}{11} = 0.\overline{72}$ Integers



Integers

The numbers you put on a number line:

Natural Numbers

Positive numbers used for counting

1, 2, 3, 4, 5, 6, 7, 8, 9, 10…

Whole Numbers

Includes all natural numbers and zero

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10…

What groups of numbers does the number belong to? 25

- Complex Number
- Rational Number
- Real Number
- Integer
- Natural Number
- Whole Number

What groups of numbers does the number belong to? $-9\frac{7}{12}$

- Complex Number
- Rational Number
- Real Number

Practice Problems

1. Let x and y be defined such that x can be any integer and y is a positive integer. Which sets of numbers must $\frac{x}{y}$ belong to?

A. Complex Numbers, Rational Numbers, Real Numbers
B. Complex Numbers, Integers, Rational Numbers
C. Integers, Natural Numbers, Whole Numbers
D. Integers, Irrational Numbers, Real Numbers
E. Complex Numbers, Integers, Natural Numbers

2. Referring to the sets of rational numbers, irrational numbers and natural numbers, which of the following statements are true about 7.77, 7/7 and 7?

A. 7.77 is rational, 7 is irrational, 7/7 is irrational
B. 7.77 is irrational, 7 is natural, 7/7 is natural
C. 7.77 is rational, 7 is rational, 7/7 is irrational
D. 7.77 is rational, 7 is natural, 7/7 is natural
E. 7.77 is rational, 7 is rational, 7/7 is irrational

3. Let $x = \sqrt{2}, y = \sqrt{3}$ and $z = \sqrt{6}$. Which of the following products results in a rational number?

A. xz
B. xy
C. yz
D. xyz
E. None of the products are rational.

4. When the expression $\frac{2\pi + 3\pi}{2\pi - 3\pi}$ is simplified, the result belongs to which groups of numbers?

A. I only
B. I and II only
C. II and III only
D. I and III only
E. I, II and III

I. Irrational Numbers
II. Integers
III. Rational Numbers

5. Let x be a natural number and let $\sqrt{x}$ be an irrational number. Which operation would NOT make a rational number?

A. Multiply the square root by itself
B. Add the square root by itself
C. Subtract the square root by itself
D. Divide the square root by itself
E. Raise the square root to the second power

6. Some sets of numbers have overlapping members. Which pair of sets *do not* have any overlapping members?

A. Complex Numbers and Integers
B. Natural Numbers and Integers
C. Irrational Numbers and Rational Numbers
D. Complex Numbers and Real Numbers
E. Rational Numbers and Integers

7. Each set below contains four members. Which set contains members that are all rational numbers?

A. $\{\,-5,\ 5\ 1/5,\ 55,\ \sqrt{55}\,\}$
B. $\{\,-20,\ -20\ 1/20,\ 0.20,\ \sqrt{20}\,\}$
C. $\{\,9,\ -9\ 1/9,\ 0.99,\ \sqrt{-9}\,\}$
D. $\{\,1,\ -1,\ 0.11,\ 1i\,\}$
E. $\{-4/9, -4.\overline{9},\ 4\ 1/9, \sqrt{4/9}\,\}$

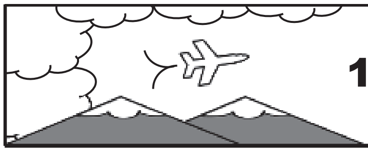
8. The number pi (π) belongs in which sets of numbers below?

A. Complex Numbers, Irrational Numbers, Real Numbers
B. Complex Numbers, Integers, Rational Numbers
C. Complex Numbers, Rational Numbers, Real Numbers
D. Integers, Irrational Numbers, Real Numbers
E. Complex Numbers, Integers, Natural Numbers

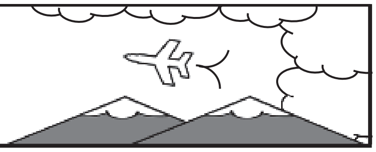
9. Which sets of numbers do the improper fraction 15/4 belong to?

A. Complex Numbers, Irrational Numbers, Real Numbers
B. Complex Numbers, Integers, Rational Numbers
C. Complex Numbers, Rational Numbers, Real Numbers
D. Integers, Irrational Numbers, Real Numbers
E. Complex Numbers, Integers, Natural Numbers

Solutions: 1A, 2D, 3D, 4C, 5B, 6C, 7E, 8A, 9C



1.2 Factors, Multiples, Primes and Prime Factorization



Factors of Numbers

Factors are whole numbers that divide evenly into a number.

You can find factors or divisors by finding all pairs that multiply to equal a number.

Factors of 36

The factors of 36 are 1,2,3,4,6,9,12,18,36

1 × 36 = 36   2 × 18 = 36   3 × 12 = 36

4 × 9 = 36   6 × 6 = 36

Factors of 88

The factors of 88 are 1,2,4,8,11,22,44,88

1 × 88 = 88   2 × 44 = 88

4 × 22 = 88   8 × 11 = 88

Prime and Composite Numbers

Prime Numbers

A prime number is a whole number bigger than 1 that only have factors of 1 and itself.

2 is prime

Factors of 2:
1 and 2
(1 × 2)

13 is prime

Factors of 13:
1 and 13
(1 × 13)

2 is the only even prime number.

Composite Numbers

Any whole number that is not prime is composite. They have factors *other* than 1 and itself.

6 is composite

Factors of 12:
1, *2*, *3*, *4*, *6*, 12
(1 × 12) (2 × 6)
(3 × 4)

25 is composite

Factors of 25:
1, *5*, 25
(1 × 25) (5 × 5)

Prime Factorization

When you write a whole number as a string of prime numbers multiplied together, it is called the **prime factorization** of the number.

Prime Factorization of 60

60 = 2 × 2 × 3 × 5
↓ ↓ ↓ ↓
ALL prime numbers ☺

Prime Factorization of 99

99 = 3 × 3 × 11
↓ ↓ ↓
ALL prime numbers ☺

The examples below are NOT prime factorizations:

60 = 1 × 2 × 2 × 3 × 5
↓
Not prime! ☹

60 = 4 × 3 × 5
↓
Not prime! ☹

Why is 1 not a prime number?

To be prime, you must have two different factors.

5 is prime.
It has two distinct factors.
(1 × 5)

1 is not prime.
It only has one factor.
(1 × 1)

Making Factor Trees

An easy way to find prime factorization is to make a factor tree.

There are 3 rules to follow when making a factor tree:

RULE 1: If a number is prime, you stop the branch.
RULE 2: If a number is not prime, you keep going.
RULE 3: You never use 1 on a factor tree.

Make a factor tree for 70

Start with two numbers that multiply to equal 70.

($70 = 7 \times 10$)

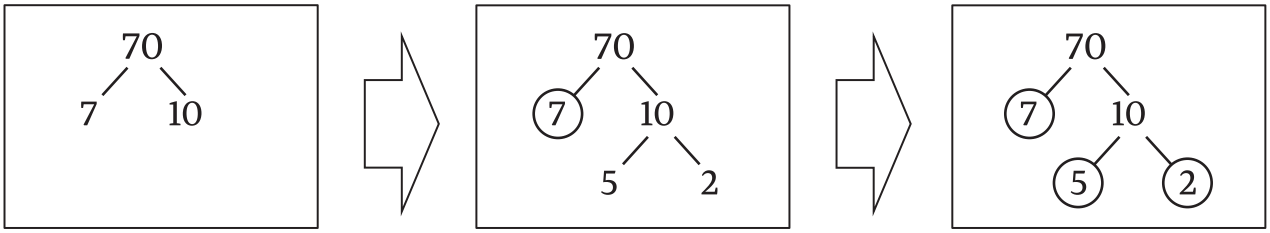
7 is prime. Stop the branch.

10 is not prime.
Keep going on that branch.

($10 = 5 \times 2$)

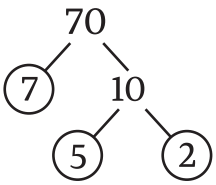
5 is prime. 2 is prime.
Stop both branches.

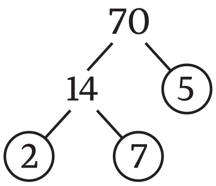
We have nowhere to go.
The tree is done.

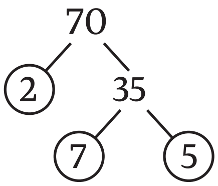


Circle your prime numbers.

There can be many ways to make a factor tree for a number.
Below are 3 different factor trees for 70.



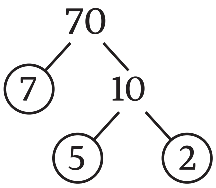




Factor Trees and Prime Factorization

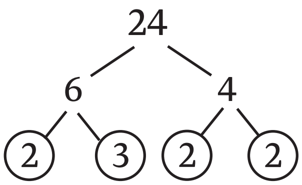
To get the prime factorization, multiply all the prime numbers on the factor tree.

Prime Factorization of 70



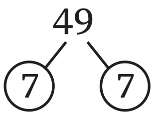
$$70 = 2 \times 5 \times 7$$
$$= 2^1 \times 5^1 \times 7^1$$

Prime Factorization of 24



$$24 = 2 \times 2 \times 2 \times 3$$
$$= 2^3 \times 3^1$$

Prime Factorization of 49



$$49 = 7 \times 7$$
$$= 7^2$$

Prime Factorization of 13

The prime factorization
of a prime number is itself. ⑬

$$13 = 13^1$$

Making Factor Tables

Another way to find prime factorization is to make a factor table.

There are 2 rules to follow when making a factor table:

RULE 1: You only divide by prime numbers.
RULE 2: You divide by prime numbers until the original number gets down to 1.

Make a factor table for 66

Draw an "L" around 66.

We will divide by primes until it gets down to 1.

$$\begin{array}{r|l} & 66 \\ \cline{2-2} \end{array}$$

11 divides evenly.
Divide by 11:

(66 ÷ **11** = 6)

$$\begin{array}{r|l} 11 & 66 \\ \cline{2-2} & 6 \end{array}$$

3 divides evenly.
Divide by 3:

(6 ÷ **3** = 2)

$$\begin{array}{r|l} 11 & 66 \\ \cline{2-2} 3 & 6 \\ \cline{2-2} & 2 \end{array}$$

2 divides evenly.
Divide by 2:

(2 ÷ **2** = 1)

$$\begin{array}{r|l} 11 & 66 \\ \cline{2-2} 3 & 6 \\ \cline{2-2} 2 & 2 \\ \cline{2-2} & 1 \end{array}$$

The factor table is done.

There can also be several ways to make a factor table for a number.
Below are 3 different factor tables for 66.

$$\begin{array}{r|l} 11 & 66 \\ \cline{2-2} 3 & 6 \\ \cline{2-2} 2 & 2 \\ \cline{2-2} & 1 \end{array} \quad \begin{array}{l} 66 \div \mathbf{11} = 6 \\ 6 \div \mathbf{3} = 2 \\ 2 \div \mathbf{2} = 1 \end{array}$$

$$\begin{array}{r|l} 2 & 66 \\ \cline{2-2} 11 & 33 \\ \cline{2-2} 3 & 3 \\ \cline{2-2} & 1 \end{array} \quad \begin{array}{l} 66 \div \mathbf{2} = 33 \\ 33 \div \mathbf{11} = 3 \\ 3 \div \mathbf{3} = 1 \end{array}$$

$$\begin{array}{r|l} 3 & 66 \\ \cline{2-2} 2 & 22 \\ \cline{2-2} 11 & 11 \\ \cline{2-2} & 1 \end{array} \quad \begin{array}{l} 66 \div \mathbf{3} = 22 \\ 22 \div \mathbf{2} = 11 \\ 11 \div \mathbf{11} = 1 \end{array}$$

Factor Tables and Prime Factorization

To get the prime factorization, multiply all the prime numbers on the left side of the table.

Prime Factorization of 66

$$\begin{array}{r|l} 11 & 66 \\ \cline{2-2} 3 & 6 \\ \cline{2-2} 2 & 2 \\ \cline{2-2} & 1 \end{array} \quad \begin{array}{l} 66 \div 11 = 6 \\ 6 \div 3 = 2 \\ 2 \div 2 = 1 \end{array}$$

$$\begin{aligned} 66 &= 2 \times 3 \times 11 \\ &= 2^1 \times 3^1 \times 11^1 \end{aligned}$$

Prime Factorization of 36

$$\begin{array}{r|l} 2 & 36 \\ \cline{2-2} 3 & 18 \\ \cline{2-2} 2 & 6 \\ \cline{2-2} 3 & 3 \\ \cline{2-2} & 1 \end{array} \quad \begin{array}{l} 36 \div 2 = 18 \\ 18 \div 3 = 6 \\ 6 \div 2 = 3 \\ 3 \div 3 = 1 \end{array}$$

$$\begin{aligned} 36 &= 2 \times 2 \times 3 \times 3 \\ &= 2^2 \times 3^2 \end{aligned}$$

Prime Factorization of 25

$$\begin{array}{r|l} 5 & 25 \\ \cline{2-2} 5 & 5 \\ \cline{2-2} & 1 \end{array} \quad \begin{aligned} 25 &= 5 \times 5 \\ &= 5^2 \end{aligned}$$

Prime Factorization of 7

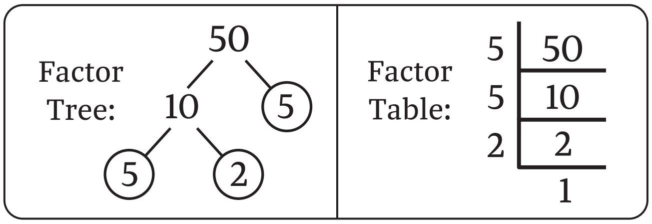
$$\begin{array}{r|l} 7 & 7 \\ \cline{2-2} & 1 \end{array} \quad 7 = 7^1$$

The prime factorization of a prime number is itself.

Summary of Prime Factorization

When you write a number as prime numbers multiplied together, it is called the prime factorization of the number.

You can find it easily by using a factor tree or a factor table.



Prime Factorization:

$$50 = 2 \times 5 \times 5 = 2^1 \times 5^2$$

Multiples of Numbers

When whole numbers are multiplied together, **multiples** are formed.

$3 \times 7 = 21$ → 21 is a multiple of 3 and 7

You can use multiplication or addition to find multiples.

Multiplying to Make Multiples

Start at the number.
Then multiply by 1, by 2, by 3, and so on.

Multiples of 4

The multiples of 4 are 4,8,12,16,24…

$4 \times 1 = 4$ $4 \times 2 = 8$ $4 \times 3 = 12$

$4 \times 4 = 16$ $4 \times 5 = 20$ $4 \times 6 = 24$

Adding to Make Multiples

Start at the number.
Then keep adding by that number.

Multiples of 5

The multiples of 5 are 5,10,15,20,25,30…

Start at *5* $5 + 5 = 10$ $10 + 5 = 15$

$15 + 5 = 20$ $20 + 5 = 25$ $25 + 5 = 30$

Practice Problems

1. What is the sum of the factors of 24?

A. 47
B. 48
C. 56
D. 58
E. 60

2. The prime factorization of 25,200 is equal to $2^A \times 3^B \times 5^C \times 7^D$. What is A + B + C + D?

A. 9
B. 10
C. 11
D. 12
E. 17

3. The number 264 has how many *distinct* prime divisors?

A. 3
B. 4
C. 5
D. 6
E. 7

4. Which number below is NOT a multiple of 12?

A. 24
B. 36
C. 60
D. 92
E. 108

5. How many distinct factors does 36 have?

A. 7
B. 8
C. 9
D. 10
E. 11

6. What is the sum of the first five prime numbers?

A. 17
B. 18
C. 28
D. 30
E. 39

7. Let A be a whole number that is divisible by 15.
Let B be a whole number that is divisible by 6.
Let AB be the product of the two numbers. If $AB < 100$, which number below *is not* a factor of AB?

A. 9
B. 10
C. 12
D. 18
E. 30

8. An "emirp" is any whole number greater than 10 that is prime when written forwards or backwards. For example,'79' is an emirp because 79 is prime and when written backwards, 97 is also prime. How many emirps are greater than 20 and less than 50?

A. 2
B. 3
C. 4
D. 5
E. 6

9. If the prime factorization of 32 is 2^5 and the prime factorization of 81 is 3^4, what is the prime factorization of the product 64×243?

A. $2^5 \times 3^5$
B. $2^6 \times 3^6$
C. $2^6 \times 3^7$
D. $2^6 \times 3^5$
E. $2^7 \times 3^7$

10. Which number below is a multiple of both 4 and 5?

A. 30
B. 40
C. 45
D. 50
E. 55

11. Let $(x + 2)$ be a multiple of 6 where $x > 0$. Which of the following is also a multiple of 6?

A. $x + 12$
B. $x + 14$
C. $x + 16$
D. $x + 20$
E. $x + 22$

12. Let A be a whole number such that its prime factorization is $7^8 \times 11^4$. What is the prime factorization of the product $A \times 154$?

A. $2^1 \times 7^9 \times 11^5$
B. $2^2 \times 7^8 \times 11^4$
C. $2^1 \times 7^8 \times 11^5$
D. $2^2 \times 7^9 \times 11^4$
E. $2^1 \times 7^9 \times 11^4$

13. What is the sum of the *second largest* factor of 39 and the *second smallest* factor of 45?

A. 16
B. 18
C. 22
D. 42
E. 84

14. When two prime numbers are added together, the result is:

I. Always odd
II. Always even
III. Always prime

A. I only
B. II only
C. II and III only
D. I and III only
E. Neither I, II nor III

Solutions: 1E, 2A, 3A, 4D, 5C, 6C, 7C, 8A, 9D, 10B, 11B, 12A, 13A, 14E

1.3 Greatest Common Factor and Least Common Multiple

Greatest Common Factor

The **Greatest Common Factor**, or GCF,
is the biggest number that divides evenly into a group of numbers.

Sometimes you can look at a group of numbers and easily work out the GCF in your head.

GCF of 6 and 8 = $\boxed{2}$

2 is the biggest number
that divides evenly into both 6 and 8.

GCF of 4 and 11 = $\boxed{1}$

1 is the only number
that divides evenly into both 4 and 11.

We will show you 3 different ways to find the GCF of a group of numbers.
Just pick the methods you like the best.

FIRST WAY: List Factors

1. List the factors of each number.
2. Pick the biggest factor they have in common.

Find the GCF of 28 and 42

Factors of 28: 1,2,4,7,*14*,28

(1 × 28) (2 × 14) (4 × 7)

Factors of 42: 1,2,3,6,7,*14*,21,42

(1 × 42) (2 × 21) (3 × 14) (6 × 7)

GCF = 14

Find the GCF of 20 and 63

Factors of 20: *1*,2,4,5,10,20

(1 × 20) (2 × 10) (4 × 5)

Factors of 63: *1*,3,7,9,21,63

(1 × 63) (3 × 21) (7 × 9)

GCF = 1

Find the GCF of 12, 24 and 36

Factors of 12: 1,2,3,4,6,*12*

(1 × 12) (2 × 6) (3 × 4)

Factors of 36: 1,2,3,4,6,9,*12*,18,36

(1 × 36) (2 × 18) (3 × 12) (4 × 9) (6 × 6)

Factors of 24: 1,2,3,4 6,8,*12*,24

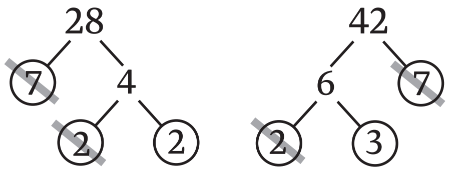
(1 × 24) (2 × 12) (3 × 8) (4 × 6)

GCF = 12

SECOND WAY: Use Factor Trees

1. Make a factor tree of each number.
2. Multiply the prime numbers that show up on ALL the trees.

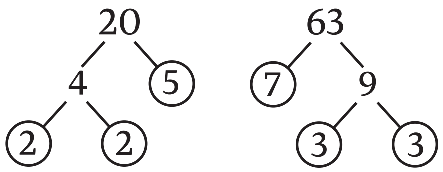
Find the GCF of 28, and 42



Both trees have a 2 and 7 in common.
Cross them off the trees.

GCF = 2 × 7 = 14

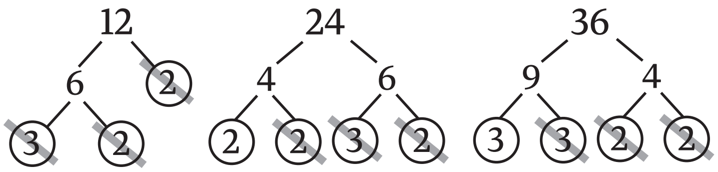
Find the GCF of 20, and 63



There are no primes that are on BOTH trees.
If they have no primes in common, the GCF will be 1.

GCF = 1

Find the GCF of 12, 24 and 36



All the trees have two 2's and one 3 in common.
Cross them off the trees.

GCF = 2 × 2 × 3 = 12

THIRD WAY: Use Factor Tables

1. Make a factor table for the group of numbers.
2. Divide until no more values divide into the numbers evenly.
3. Multiply the prime numbers on the left side of the table to get the GCF.

Find the GCF of 28 and 42

Draw an "L" around 28 and 42.

Divide until no more numbers divide into 28 and 42 evenly.

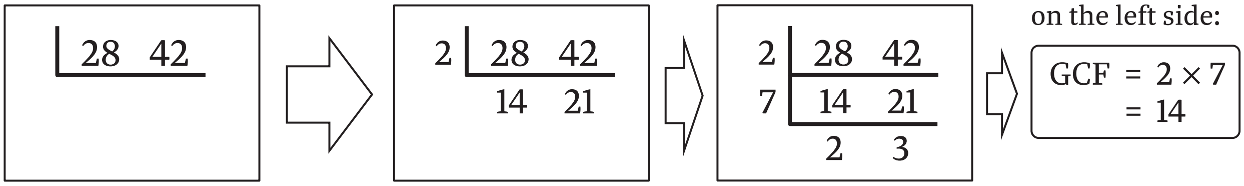
2 divides evenly.
Divide by 2:

$(28 \div \mathbf{2} = 14)$
$(42 \div \mathbf{2} = 21)$

7 divides evenly.
Divide by 7:

$(14 \div \mathbf{7} = 2)$
$(21 \div \mathbf{7} = 3)$

Multiply the numbers on the left side:



Nothing divides into both 2 and 3. We are done.

Find the GCF of 20 and 63

$$\begin{array}{|cc} 20 & 63 \\ \hline \end{array}$$

There is no number that divides into both 20 and 63 at the same time except 1.

The factor table is done.

When there is no number to divide by, the GCF will be 1.

GCF = 1

Find the GCF of 12, 24 and 36

Below is a factor table for the three numbers:

$$\begin{array}{r|rrr} 2 & 12 & 24 & 36 \\ \hline 2 & 6 & 12 & 18 \\ \hline 3 & 3 & 6 & 9 \\ \hline & 1 & 2 & 3 \end{array}$$

Multiply the numbers on the left side:

GCF = 2 × 2 × 3 = 12

$$\frac{1}{2} + \frac{3}{5} \quad \Downarrow \quad \frac{5}{10} + \frac{6}{10}$$

2 and 5:
LCM = 10

Least Common Multiple

The **Least Common Multiple**, or LCM,
is the smallest multiple that a group of numbers have in common.

Most students use the LCM when making common denominators.

We will also show three different ways to find the LCM.

$$\frac{3}{4} - \frac{2}{8} \quad \Downarrow \quad \frac{6}{8} - \frac{2}{8}$$

4 and 8:
LCM = 8

FIRST WAY: List Multiples

1. List the multiples of each number.
2. Pick the first multiple they have in common.

Find the LCM of 6 and 9

Multiples of 6:
6, 12, *18*, 24, 30, 36...

Multiples of 9:
9, *18*, 27, 36, 45, 54...

⇨ LCM = 18

Find the LCM of 3 and 12

Multiples of 3:
3, 6, 9, *12*, 15, 18...

Multiples of 12:
12, 24, 36, 48...

⇨ LCM = 12

Find the LCM of 3, 5 and 6

Multiples of 3:
3, 6, 9, 12, 15, 18, 21, 24, 27, *30*...

Multiples of 5:
5, 10, 15, 20, 25, *30*...

Multiples of 6:
6, 12, 18, 24, *30*...

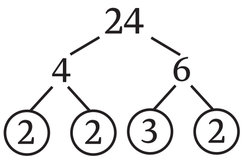
⇨ LCM = 30

SECOND WAY: Use Prime Factors

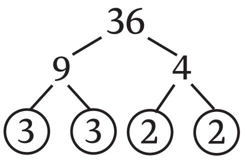
1. Write the prime factorization of each number using exponents.
2. Multiply by the highest power of each prime number you see.

Find the LCM of 24, and 36

Get the prime factorization of each number:



$24 = 2^3 \times 3^1$



$36 = 2^2 \times 3^2$

$24 = 2^3 \times 3^1 \qquad 36 = 2^2 \times 3^2$

The prime numbers are 2 and 3.

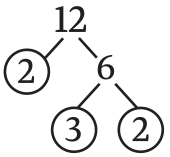
The highest power of the 2s is THREE ⇨ 2^3

The highest power of the 3s is TWO ⇨ 3^2

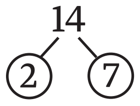
$$\text{LCM} = 2^2 \times 3^2 = 72$$

Find the LCM of 12, 14 and 28

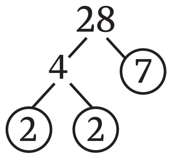
Get the prime factorization of each number:



$12 = 2^2 \times 3^1$



$14 = 2^1 \times 7^1$



$24 = 2^2 \times 7^1$

$12 = 2^2 \times 3^1 \qquad 14 = 2^1 \times 7^1 \qquad 28 = 2^2 \times 7^1$

The prime numbers are 2, 3 and 7.

The highest power of the 2s is TWO ⇨ 2^2

The highest power of the 3s is ONE ⇨ 3^1

The highest power of the 7s is ONE ⇨ 7^1

$$\text{LCM} = 2^2 \times 3^1 \times 7^1 = 84$$

THIRD WAY: Use Factor Tables

1. Make a factor table for the group of numbers.
2. Divide until no more values divide into the numbers evenly.
3. Multiply ALL the numbers around the outside to get the LCM.

Find the LCM of 12 and 30

Draw an "L" around 12 and 18.

Divide until no more numbers divide into 12 and 18 evenly.

$$\begin{array}{r|cc} & 12 & 30 \\ \cline{2-3} \end{array}$$

3 divides evenly.
Divide by 3:
(12 ÷ **3** = 4)
(30 ÷ **3** = 10)

$$\begin{array}{r|cc} 3 & 12 & 30 \\ \cline{2-3} & 4 & 10 \end{array}$$

2 divides evenly.
Divide by 2:
(4 ÷ **2** = 2)
(10 ÷ **2** = 5)

$$\begin{array}{r|cc} 3 & 12 & 30 \\ \cline{2-3} 2 & 4 & 10 \\ \cline{2-3} & 2 & 5 \end{array}$$

Nothing divides into both 2 and 5. We are done.

Multiply all the numbers on the outside.

LCM
= 3 × 2 × 2 × 5
= 60

Find the LCM of 3 and 11

$$\begin{array}{r|cc} & 3 & 11 \\ \cline{2-3} \end{array}$$

There is no number that divides into both 3 and 11 at the same time.

The factor table is done.

Just multiply the numbers in the factor table to get the LCM:

LCM = 3 × 11 = 33

Find the LCM of 24 and 36

Below is a factor table for the two numbers:

$$\begin{array}{r|cc} 2 & 24 & 36 \\ \cline{2-3} 3 & 12 & 18 \\ \cline{2-3} 2 & 4 & 6 \\ \cline{2-3} & 2 & 3 \end{array}$$

Multiply the numbers on the outside:

LCM = 2 × 3 × 2 × 2 × 3 = 72

Special note about the LCM and factor tables:

The LCM method for factor tables only works for 2 numbers.

<u>If you want to find the LCM of 3 numbers using factor tables:</u>
1. Find the LCM of the first two numbers.
2. Use the answer to find the LCM with the 3rd number.

Find the LCM of 18, 27 and 42

<u>Find the LCM of 18 and 27:</u>

$$\begin{array}{r|cc} 3 & 18 & 27 \\ \cline{2-3} 3 & 6 & 9 \\ \cline{2-3} & 2 & 3 \end{array}$$

LCM = 3 × 3 × 2 × 3 = 54

<u>Now find the LCM of 54 and 42:</u>

$$\begin{array}{r|cc} 2 & 54 & 42 \\ \cline{2-3} 3 & 27 & 21 \\ \cline{2-3} & 9 & 7 \end{array}$$

LCM = 2 × 3 × 9 × 7 **=** $\boxed{378}$

Least Common Denominator

The **Least Common Denominator**, or LCD,
is the least common multiple of the denominators.

Find the LCD: $\frac{5}{6}, \frac{4}{9}, \frac{7}{12}$

The denominators are 6, 9 and 12.

We will list multiples to get the answer:

Multiples of 6 ⇨ 6, 12, 18, 24, 30, *36*…

Multiples of 9 ⇨ 9, 18, 27, *36*, 45…

Multiples of 12 ⇨ 12, 24, *36*, 48, 60…

LCD = 36

The example above was straightforward.
To the right is a much harder example of the LCD.

Find the LCD: $\frac{3}{8}, \frac{19}{35}, \frac{17}{50}$

The denominators are 8, 35 and 50.

We will use prime factors to get the answer:

$8 = 2^3 \qquad 35 = 5^1 \times 7^1 \qquad 50 = 2^1 \times 5^2$

The prime numbers are 2, 5 and 7.

The highest power of the 2s is THREE ⇨ 2^3

The highest power of the 5s is TWO ⇨ 5^2

The highest power of the 7s is ONE ⇨ 7^1

$\text{LCD} = 2^3 \times 5^2 \times 7^1 = 1400$

Mixing Up Factors and Multiples

Students mix up these concepts often.
Below we show examples of factors, multiples, the GCF and LCM.

Find the factors and multiples of 12

Factors divide evenly into a number:

Factors of 12 = 1,2,3,4,6,12

Multiples are found by skip counting:

Multiples of 12 = 12,24,36,48,60…

Find the GCF and LCM of 5 and 10

Factors of 5:
1, *5*

Factors of 10:
1, 2, *5*, 10

GCF = 5

Multiples of 5:
5, *10*, 15, 20…

Multiples of 10:
10, 20, 30, 40…

LCM = 10

Find the GCF and LCM of 6 and 9

Factors of 6:
1, 2, *3*, 6

Factors of 9:
1, *3*, 9

GCF = 3

Multiples of 6:
6, 12, *18*, 24…

Multiples of 9:
9, *18*, 27, 36…

LCM = 18

Practice Problems

1. Let x be the GCF of 30 and 15. Let y be the LCM of 20 and 10. What is the value of $x - y$?

A. –5
B. 5
C. 20
D. –20
E. 15

2. One of the pairs of numbers below have a GCF of 3 and an LCM of 45. Which pair is it?

A. 3, 15
B. 5, 15
C. 6, 15
D. 9, 15
E. 12, 15

3. What is the greatest common factor of 36, 42 and 90?

A. 6
B. 9
C. 18
D. 630
E. 1260

4. What is least common denominator of 8/15, 2/25 and 1/10?

A. 5
B. 60
C. 75
D. 150
E. 300

5. What is the largest number that divides into the numerator and denominator to reduce the fraction 1360/1480?

A. 10
B. 20
C. 40
D. 80
E. 160

6. A radio station gives out cash prizes. A cash prize of $1,000 is given out every 15 minutes, $500 is given out every 12 minutes and $250 is given out every 40 minutes. If the radio station gave out all three prizes at the same time at 8:00am, what time will it be the next time that all three prizes are given out at the same time?

A. 9:00am
B. 10:00am
C. 10:30am
D. 11:00am
E. 11:30am

7. What is the GCF of 56, 72 and 75?

A. 1
B. 3
C. 4
D. 5
E. 8

8. Which of the following sets of numbers have a greatest common factor of 6 and a least common multiple of 72?

A. 12, 16
B. 12, 18
C. 16, 18
D. 16, 24
E. 18, 24

9. What is least common denominator of 1/14, 7/12 and 9/10?

A. 120
B. 240
C. 360
D. 420
E. 630

10. Two natural numbers are "coprime" if the only whole number that divides evenly into both numbers at the same time is 1. Which pair of numbers are coprime?

A. 26, 39
B. 27, 56
C. 34, 58
D. 35, 91
E. 42, 51

11. When $\frac{a}{6} + \frac{b}{15} - \frac{c}{75}$ is solved, which of the following will be the common denominator?

A. 150
B. 200
C. 225
D. 250
E. 350

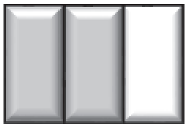
Solutions: 1A, 2D, 3A, 4D, 5C, 6B, 7A, 8E, 9D, 10B, 11A

1.4 Fractions and Mixed Numbers

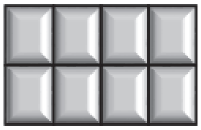
Overview of Fractions and Mixed Numbers

Fractions are numbers that show parts of a whole.

$\frac{2}{3}$ → Numerator → Denominator



$\frac{8}{8}$ = 1 Whole

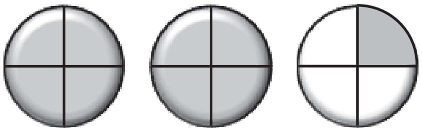


Mixed numbers are a sum of the number of wholes and the parts of a whole.

Any mixed number can be written as an improper fraction.

Improper fractions are fractions where the numerator is equal to or bigger than the denominator.

What improper fraction and mixed number is shown in the picture below?



The circles are cut into fourths and 9 of the parts are shaded ⇨ $\frac{9}{4}$

We have 2 Whole circles and 1/4 of another circle ⇨ $2\frac{1}{4}$

Changing Between Improper Fractions and Mixed Numbers

Changing Mixed Numbers to Improper Fractions

Multiply the whole number by the denominator. Then add that answer to the numerator.
The denominator stays the same.

$$2\frac{1}{4} \Rightarrow 2_{\times}\frac{1+}{4} \Rightarrow \boxed{\frac{9}{4}}$$

2 × 4 + 1 = 9

$$3\frac{5}{6} \Rightarrow 3_{\times}\frac{5+}{6} \Rightarrow \boxed{\frac{23}{6}}$$

3 × 6 + 5 = 23

Changing Improper Fractions to Mixed Numbers

Divide to change an improper fraction to a mixed number.
The remainder is the new numerator. The denominator stays the same.

$$\frac{9}{4} \Rightarrow \begin{array}{r} 2 \\ 4\overline{)\,9} \\ -\,8 \\ \hline 1 \end{array} \quad \begin{array}{l} 9 \div 4 \\ = 2R1 \end{array} \Rightarrow \boxed{2\frac{1}{4}}$$

$$\frac{47}{8} \Rightarrow \begin{array}{r} 5 \\ 8\overline{)\,47} \\ -\,40 \\ \hline 7 \end{array} \quad \begin{array}{l} 47 \div 8 \\ = 5R7 \end{array} \Rightarrow \boxed{5\frac{7}{8}}$$

Math Operations of Fractions and Mixed Numbers

Even though you will have a calculator for the ACT test,
it is still good to know how do all math operations of fractions by hand.

Adding and Subtracting Fractions and Mixed Numbers

When adding and subtracting fractions, make sure they have the same denominator.

ADD: $\frac{1}{4} + \frac{3}{6}$

STEP 1: Find a common denominator.

The denominators are 4 and 6.

Multiples of 4 ⇨ 4, 8, *12*, 16, 20, 24...

Multiples of 6 ⇨ 6, *12*, 18, 24, 30, 36...

LCD = 12

STEP 2: Multiply to find the numerators.

$$\frac{1^{\times 3}}{4_{\times 3}} = \frac{\boxed{3}}{12} \qquad \frac{3^{\times 2}}{6_{\times 2}} = \frac{\boxed{6}}{12}$$

STEP 3: Add the fractions.

$$\frac{3}{12} + \frac{6}{12} = \frac{9}{12} \xrightarrow{\text{Reduce}} \boxed{\frac{3}{4}}$$

SUBTRACT: $1 - \frac{2}{7}$

STEP 1: Find a common denominator.

$$1 - \frac{2}{7} \xrightarrow{\text{SAME AS}} \frac{1}{1} - \frac{2}{7}$$

The denominators are 1 and 7.

LCD = 7

STEP 2: Multiply to find the numerators.

$$\frac{1^{\times 7}}{1_{\times 7}} = \frac{\boxed{7}}{7} \qquad \frac{2^{\times 1}}{7_{\times 1}} = \frac{\boxed{2}}{7}$$

STEP 3: Subtract the fractions.

$$\frac{7}{7} - \frac{2}{7} = \boxed{\frac{5}{7}}$$

For 1/4 + 3/6, we could have also used 24 as a common denominator.

Using the LCD gives you smaller numbers to work with and less work to do.

To add and subtract mixed numbers, first change the mixed numbers to improper fractions.
Then add or subtract the fractions.

SUBTRACT: $3\frac{1}{2} - 1\frac{4}{5}$

$$3\frac{1}{2} - 1\frac{4}{5}$$

Improper Fractions

$$\frac{7}{2} - \frac{9}{5}$$

$$\begin{array}{rl} & \frac{7^{\times 5}}{2_{\times 5}} = \frac{\boxed{35}}{10} \\ - & \frac{9^{\times 2}}{5_{\times 2}} = \frac{\boxed{18}}{10} \\ \hline & \qquad\ \ \frac{17}{10} \Rightarrow \boxed{1\frac{7}{10}} \end{array}$$

LCD = 10

Multiplying Fractions and Mixed Numbers

To multiply fractions, multiply the numerators and multiply the denominators.

MULTIPLY: $\frac{2}{6} \times \frac{4}{5}$

$$\frac{2}{6} \times \frac{4}{5} = \frac{8}{30} \xrightarrow{\text{Reduce}} \boxed{\frac{4}{15}}$$

MULTIPLY: $\frac{2}{17} \times 6$

$$\frac{2}{17} \times \frac{6}{1} = \boxed{\frac{12}{17}}$$

To multiply mixed numbers, first change the mixed numbers to improper fractions.
Then multiply the fractions.

MULTIPLY:
$2\frac{3}{4} \times 1\frac{1}{2}$

$$2\frac{3}{4} \times 1\frac{1}{2}$$

Improper Fractions

$$\frac{11}{4} \times \frac{3}{2}$$

$$\frac{11}{4} \times \frac{3}{2} = \frac{33}{8} \Rightarrow \boxed{4\frac{1}{8}}$$

Dividing Fractions and Mixed Numbers

To divide fractions, you follow K.C.F.
Keep the first fraction, change the sign to multiplication, and flip the second fraction.

DIVIDE:
$\frac{2}{7} \div \frac{3}{5}$

$$\frac{2}{7} \div \frac{3}{5}$$

K C F

$$\frac{2}{7} \times \frac{5}{3} = \boxed{\frac{10}{21}}$$

DIVIDE:
$\frac{1}{2} \div 9$

$$\frac{1}{2} \div \frac{9}{1}$$

K C F

$$\frac{1}{2} \times \frac{1}{9} = \boxed{\frac{1}{18}}$$

To divide mixed numbers, first change the mixed numbers to improper fractions.
Then divide the fractions.

DIVIDE:
$2\frac{1}{2} \div 1\frac{2}{6}$

$$2\frac{1}{2} \div 1\frac{2}{6}$$

Improper Fractions

$$\frac{5}{2} \div \frac{8}{6}$$

$$\frac{5}{2} \div \frac{8}{6}$$

$$\frac{5}{2} \times \frac{6}{8} = \frac{30}{16} \Rightarrow \boxed{1\frac{7}{8}}$$

NOTE:
$\frac{30}{16} = 1\frac{14}{16} = 1\frac{7}{8}$

Finding Fractions of Amounts

Finding fractions or parts of groups is a concept that shows up regularly on the test.

To find the fraction of any amount, *multiply* the fraction by the amount.

Twelve t-shirts are washed, folded and placed in a drawer.
If 2/3 of the t-shirts were black, how many black t-shirts were placed in the drawer?

Find 2/3 of 12:

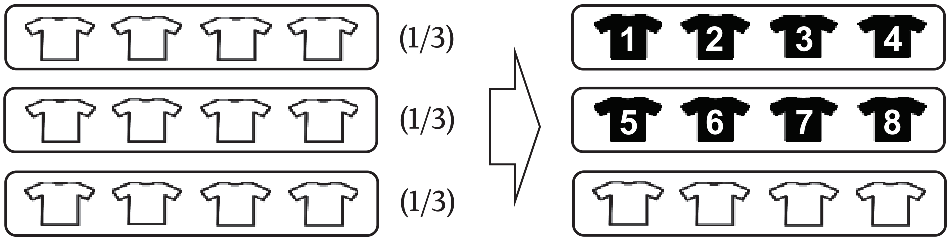
8 t-shirts are black.

$$\frac{2}{3} \times 12 \Rightarrow \frac{2}{3} \times \frac{12}{1} = \frac{24}{3} = \boxed{\text{8 t-shirts}}$$

We can show using pictures that 2/3 of 12 = 8:

Draw 12 t-shirts. Divide them into thirds.
(3 equal groups)

Now count the number of t-shirts
in two of the three groups:



In 2/3 of the groups,
there are 8 t-shirts.

$$\frac{2}{3} \text{ of } 12 = 8$$

At a picnic, 3/4 of all the people who attended are kids. One third of all the kids are girls.
How many girls were at the picnic if 60 people attended the picnic?

We can solve the problem two different ways.

FIRST WAY:

Find the number of kids.

3/4 of 60 people:

$$\frac{3}{4} \times 60 = 45 \text{ kids}$$

Now find the number of girls.

1/3 of 45 kids:

$$\frac{1}{3} \times 45 = \boxed{\text{15 girls}}$$

SECOND WAY:

Find the number of girls in one step.

Find 1/3 of 3/4 of 60 people:

$$\frac{1}{3} \times \frac{3}{4} \times 60 = \frac{180}{12} = \boxed{\text{15 girls}}$$

Word Problems Involving One Whole

Sometimes you may have to subtract from one whole and find out what fraction is left over.

Paula ate 5/9 of a pizza.
What fraction of the pizza did she not eat?

The entire pizza represents 1 whole.
We can subtract to get the pizza she did not eat.

4/9 of the pizza was not eaten.

$$\underset{\text{Whole Pizza}}{1} - \underset{\text{Pizza Eaten}}{\frac{5}{9}} \Rightarrow \frac{9}{9} - \frac{5}{9} = \underset{\text{Pizza left over}}{\boxed{\frac{4}{9}}}$$

It makes sense. If she ate 5 of the 9 slices, four slices will be left over (9 – 5 = 4).

Joey just received his paycheck. He spent 2/5 of his paycheck on rent.
He spent another 1/4 of the check on food. What fraction of the paycheck is left over?

There are two different ways to solve the problem.

FIRST WAY:
Find the total and then subtract from 1 whole.

Find the fraction of the check that was spent:

$$\underset{\text{Rent}}{\frac{2}{5}} + \underset{\text{Food}}{\frac{1}{4}} \Rightarrow \frac{8}{20} + \frac{5}{20} = \underset{\text{Spent}}{\frac{13}{20}}$$

The entire paycheck is <u>1 whole</u>.
Subtract off what is spent to find out what is left over.

$$\underset{\text{Whole Paycheck}}{1} - \underset{\text{Spent}}{\frac{13}{20}} \Rightarrow \frac{20}{20} - \frac{13}{20} = \boxed{\frac{7}{20}}$$

SECOND WAY:
Subtract all the fractions from 1 whole.

Subtract 2/5 of a check and 1/4 of a check from 1 whole paycheck:

$$\underset{\text{Whole Paycheck}}{1} - \underset{\text{Rent}}{\frac{2}{5}} - \underset{\text{Food}}{\frac{1}{4}} \Rightarrow \frac{20}{20} - \frac{8}{20} - \frac{5}{20} = \boxed{\frac{7}{20}}$$

7/20 of the paycheck is left over.

Finding the Halfway Point Between Fractions

To find the halfway point between any two numbers, you add the numbers and divide by 2.
In other words, you find the average of the two numbers.

What number is halfway between 2 and 4?

The answer is 3.
Most people can figure it out in their head.

$$\Rightarrow \frac{2+4}{2} = \frac{6}{2} = \boxed{3}$$

What number is halfway between 4/9 and 1/2?

This problem is much harder to solve.
We will add the fractions and then divide the answer by 2.

Add the fractions:

$$\frac{4}{9} + \frac{1}{2} \Rightarrow \frac{8}{18} + \frac{9}{18} = \frac{17}{18}$$

Now divide by 2:

$$\frac{17}{18} \div \frac{2}{1} \Rightarrow \frac{17}{18} \times \frac{1}{2} = \boxed{\frac{17}{36}}$$

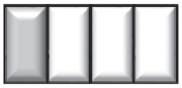
Minimizing and Maximizing Fractions

Some problems require you to make a fraction as large or as small as possible.

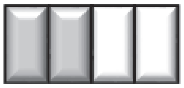
Making Fractions Bigger

As numerators get bigger, fractions get *bigger*.

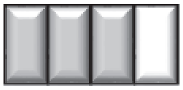
$\frac{\mathbf{1}}{4}$



$\frac{\mathbf{2}}{4}$



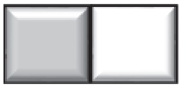
$\frac{\mathbf{3}}{4}$



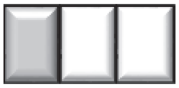
Making Fractions Smaller

As denominators get bigger, fractions get *smaller*.

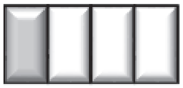
$\frac{1}{\mathbf{2}}$



$\frac{1}{\mathbf{3}}$



$\frac{1}{\mathbf{4}}$



To make the biggest fraction possible: ⇨ $\dfrac{\text{Big Numerator}}{\text{Small Denominator}}$

To make the smallest fraction possible: ⇨ $\dfrac{\text{Small Numerator}}{\text{Big Denominator}}$

If x is a number in the range $1 \le x \le 9$ and y is a number in the range $5 \le y \le 13$, what is the largest possible value for the fraction $\dfrac{x}{y}$?

The biggest possible fraction will have the biggest numerator and smallest denominator:

$$\frac{x}{y} \quad \begin{matrix} \Rightarrow \text{Biggest x} \rightarrow 9 \\ \Rightarrow \text{Smallest y} \rightarrow 5 \end{matrix} \quad \Rightarrow \boxed{\frac{9}{5}}$$

Comparing Fractions and Decimals

Before covering this topic, you need to understand terminating and repeating decimals.

Terminating Decimals

Terminating decimals have a stopping point.

Adding zeros to the end of terminating decimals will not change their value.

$0.4 = 0.4\mathbf{0} = 0.4\mathbf{00}$

$0.26 = 0.26\mathbf{0} = 0.26\mathbf{00}$

$0.197 = 0.197\mathbf{0} = 0.197\mathbf{00}$

Repeating Decimals

Repeating decimals repeat forever in a pattern.

We use a bar or a line to show the group of digits that repeat forever.

Decimal repeats in 4s: $0.\overline{4} = 0.444444\ldots$

Decimal repeats in 83s: $0.\overline{83} = 0.838383\ldots$

The easiest way to compare fractions and decimals is to use your calculator to change everything to decimals.

(For example, to change 3/4 to a decimal, put 3 ÷ 4 in your calculator)

List from least to greatest: $0.\overline{52}$, 1/2, $0.\overline{5}$, 11/20, 0.4999

Give all the decimals the same number of decimal places to make it easier to rank them.

Change everything to decimals:

$0.\overline{52} \Rightarrow 0.52\mathbf{52}$

$1/2 = 0.5 \Rightarrow 0.5\mathbf{000}$

$0.\overline{5} \Rightarrow 0.5\mathbf{555}$

$11/20 = 0.55 \Rightarrow 0.55\mathbf{00}$

$0.4949 \Rightarrow 0.4949$

Now rank the decimals:

0.4949 ⇨ 0.5000 ⇨ 0.5252 ⇨ 0.5500 ⇨ 0.5555

$$\boxed{\underset{\text{Least}}{0.4949} < 1/2 < 0.\overline{52} < 11/20 < \underset{\text{Greatest}}{0.\overline{5}}}$$

List from least to greatest: 4/9, 5/6, 2/3

Change everything to decimals and rank them:

4/9 ⇨ 0.444

5/6 ⇨ 0.833

2/3 ⇨ 0.667

0.444 ⇨ 0.667 ⇨ 0.833

$$\boxed{\underset{\text{Least}}{4/9} < 2/3 < \underset{\text{Greatest}}{5/6}}$$

You can also use a common denominator as long as you can find it quickly.

$$\frac{4^{\times 2}}{9_{\times 2}} = \frac{\boxed{8}}{18} \qquad \frac{2^{\times 6}}{3_{\times 6}} = \frac{\boxed{12}}{18} \qquad \frac{5^{\times 3}}{6_{\times 3}} = \frac{\boxed{15}}{18}$$

Practice Problems

1. Let x be a factor of 24 and let y be a factor of 36. How many fractions in the form of $\frac{x}{y}$ can be made that are equivalent to 5/10?

A. 3
B. 4
C. 5
D. 6
E. 7

2. Let x and y be positive integers under the conditions of $6 < x < 16$ and $6 < y < 16$. What is the largest possible value of $\frac{x-3}{y+5}$?

A. 1
B. 4/3
C. 8/3
D. 12/11
E. 13/11

3. If w, y and z are positive integers, which of the following statements is false?

A. $\frac{w+y}{z} = \frac{w}{z} + \frac{y}{z}$

B. $\frac{w+y}{w} = 1 + \frac{y}{w}$

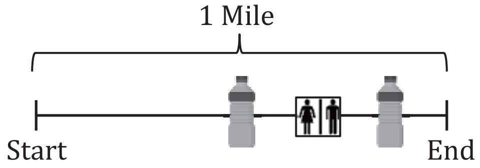
C. $\frac{w+y}{w} = y$

D. $\frac{w-y}{y} = \frac{w}{y} - 1$

E. $\frac{wy}{w} = y$

4. A charity is putting together a one–mile walk. Two water stations were placed exactly half a mile away and 7/8 of a mile away from the starting point of the race, respectively. If a bathroom station is placed halfway between the water stations, how far away in miles will it be from the starting line?

A. 3/4
B. 5/8
C. 7/8
D. 11/16
E. 13/16



5. Which of the following fractions is equivalent to the expression below?

A. 10/21
B. 4/25
C. 10/27
D. 10/33
E. 4/35

$$\boxed{\frac{\frac{1}{2}+\frac{1}{3}}{2-\frac{1}{4}}}$$

6. Madison drinks 3/7 of an entire container of juice. She equally distributes the left-over juice into 5 cups. What fraction of the original container of juice is contained in each cup?

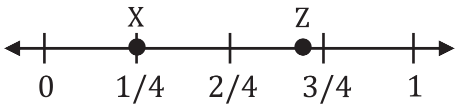
A. 3/245
B. 4/35
C. 6/35
D. 8/35
E. 3/245

7. There are 1320 students that go to a high school. If 3/8 of all the students are freshmen and 2/11 of all the students are seniors, what is the combined number of freshmen and seniors that attend the high school?

A. 90
B. 348
C. 585
D. 735
E. 855

8. Points X and Z denote the approximate locations of 1/4 and 8/11 on the number line below. If Point Y is equidistant from points X and Z, what fraction is located at point Y?

A. 5/11
B. 13/22
C. 11/24
D. 23/44
E. 43/88



9. On one particular day of a college class, 1/3 of all the students were out sick and 2/7 of all the students arrived to class late. The rest of the students arrived to class on time. What fraction of the students in the college class arrived to class on time?

A. 3/10
B. 7/10
C. 8/21
D. 13/21
E. 16/21

10. Let m be a multiple of 3 in the range $10 \leq m \leq 20$ and let p be a prime number in the range $20 \leq p \leq 30$. What is the smallest possible value of $\frac{m}{p}$?

A. 12/23
B. 12/29
C. 15/23
D. 15/29
E. 18/29

11. In a parking lot that contains a total of 60 cars, 4/15 of the cars are black, 1/4 of the cars are white, and 7/20 of the cars are gray. How many cars in the parking lot are *not* black, white or gray?

A. 8
B. 10
C. 12
D. 14
E. 16

12. Which choice below shows the correct order of the numbers 33/50, 0.65599, $0.\overline{6}$ and $0.\overline{65}$ from least to greatest?

A. $0.65599 < 0.\overline{65} < 33/50 < 0.\overline{6}$
B. $33/50 < 0.65599 < 0.\overline{6} < 0.\overline{65}$
C. $0.\overline{65} < 0.65599 < 0.\overline{6} < 33/50$
D. $0.65599 < 0.\overline{65} < 0.\overline{6} < 33/50$
E. $33/50 < 0.\overline{6} < 0.65599 < 0.\overline{65}$

13. Which of the numbers below is the smallest?

A. 3/8
B. 2/5
C. $0.37\overline{4}$
D. 234/625
E. 19/50

14. Which of the following fractions is equivalent to the expression below?

$$\boxed{\frac{2\frac{1}{3} + 1\frac{5}{6}}{3}}$$

A. 13/9
B. 14/9
C. 19/18
D. 23/18
E. 25/18

15. A man bought small, medium and large-sized boxes from a home improvement store. Of the boxes he bought, 1/12 of them were large-sized boxes and 1/8 of them were medium-sized boxes. What fraction of the total boxes bought were small-sized boxes?

A. 7/8
B. 7/12
C. 9/10
D. 19/24
E. 23/24

16. Maxine, Laura and John are splitting a 4-pound bag of candy. Maxine took 1 1/3 pounds of candy out of the bag for herself and John took 1 1/4 pounds of candy out of the bag for himself. The rest of the candy in the bag went to Laura.

Which of the following gives the ranking in ascending order of the pounds of candy the three people took out of the bag?

A. John < Maxine < Laura
B. Maxine < John < Laura
C. John < Laura < Maxine
D. Maxine < Laura < John
E. Laura < John < Maxine

17. Kala is making a chicken recipe. One serving of the recipe requires 3/4 of a pound of chicken. If the grocery store she visits sells chicken for $1.30 per pound, how many servings of the recipe can she make with $11.70? (You can ignore sales tax with this problem.)

A. 9
B. 11
C. 12
D. 13
E. 15

18. There are 35 adults in a room. Four–sevenths of the adults are lawyers. If 2/5 of the lawyers are female, how many female lawyers were in the room?

A. 8
B. 10
C. 12
D. 14
E. 16

19. Within a 2–hour span, a student spent 1/8 of the time taking a shower, 1/4 of the time eating dinner and spent another 4/15 of the time doing math homework. The rest of the leftover time was used to talk to a friend on the phone. How many *minutes* did the person spend talking on the phone?

A. 27
B. 43
C. 51
D. 69
E. 77

20. Given the group of fractions {5/6, 4/7, 3/14, 1/2}, how much larger is the biggest fraction in the group compared to the smallest fraction in the group?

A. 1/3
B. 1/14
C. 2/7
D. 5/14
E. 13/21

21. Which choice below shows the correct order of the numbers 4/5, $0.7\overline{9}$, 799/1000 and $0.\overline{79}$ from least to greatest?

A. $0.7\overline{9} < 0.\overline{79} < 799/1000 < 4/5$
B. $0.\overline{79} < 0.7\overline{9} < 799/1000 < 4/5$
C. $0.7\overline{9} < 0.\overline{79} < 4/5 < 799/1000$
D. $0.\overline{79} < 799/1000 < 0.7\overline{9} < 4/5$
E. $79/1000 < 0.7\overline{9} < 0.\overline{79} < 4/5$

22. What is the value of X in the equation below?

$$\boxed{X + 3\frac{2}{7} + 2\frac{1}{3} = 7\frac{1}{7}}$$

A. 1 8/21
B. 1 10/21
C. 1 11/21
D. 2
E. 2 2/21

23. A store has 2,000 pieces of fruit for sale in their produce section. If 3 out of every 20 pieces of fruit are apples, and 7 out of every 12 apples are Gala apples, how many Gala apples are available for sale in the store?

A. 140
B. 150
C. 160
D. 175
E. 195

24. A rectangular sheet of steel that is 14 2/3 feet wide is getting cut into strips that are 1 5/6 feet wide. How many strips can be cut from the original sheet of steel?

A. 7
B. 8
C. 9
D. 10
E. 11

25. What is the answer to the problem below?

$$\left(1\frac{4}{13} \bullet \frac{3}{17}\right) + \left(\frac{4}{23} \bullet 1\frac{10}{13}\right) + \left(2\frac{3}{13} \bullet \frac{6}{29}\right)$$

A. 1
B. 1 1/2
C. 2
D. 2 1/2
E. 3

26. Set X {3, 5, 7, 9} and Set Y {2, 4, 6, 8} each contain four positive integers. Let a and b be members from Set X and let c and d be members from Set Y. What is the smallest possible answer to $\frac{a}{c} \bullet \frac{d}{b}$?

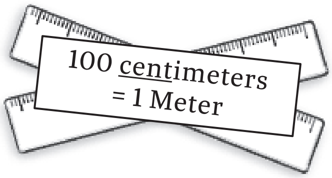
A. 1/9
B. 1/12
C. 1/16
D. 4/81
E. 9/64

Solutions: 1B, 2A, 3C, 4D, 5A, 6B, 7D, 8E, 9C, 10B, 11A, 12A, 13D, 14E, 15D, 16A, 17C, 18A, 19B, 20E, 21D, 22C, 23D, 24B, 25A, 26B

1.5 Percent Problems

Overview of Percents

The word "per<u>cent</u>" means "per 100".



$50\% = \frac{50}{100} = \frac{1}{2}$

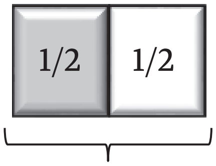
An item is on sale for 1/2 off
or 50% off.

PERCENT
$X\% = \frac{X}{100}$

$100\% = \frac{100}{100} = 1$

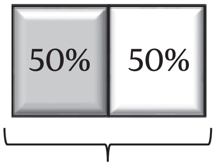
A man ate 1 whole pizza
or 100% of the pizza.

Fractions, decimals and percents are all just 3 different ways of showing parts of a whole.



Entire Box
= 1 Whole

Fractions		Decimals		Percents
$\frac{1}{2}$	=	0.50	=	50%
Fractions use a "fraction bar"		Decimals use a "decimal point"		Percents use a "percent sign"



Entire Box
= 100%

Fraction, Decimal and Percent Conversions

In order to do well on the ACT,
you need to be able to switch quickly between fractions, decimals and percents.

Changing Fractions to Decimals

Normally, in order to change fractions to decimals, we would need to use long division.
This will use up too much time on the test.

To change fractions to decimals, use the calculator to divide in the *exact order* it is written.

Change to a decimal: 4/5

$\frac{4}{5} \Rightarrow 4 \div 5 \Rightarrow \boxed{0.8}$

Change to a decimal: 5/4

$\frac{5}{4} \Rightarrow 5 \div 4 \Rightarrow \boxed{1.25}$

Changing Decimals to Fractions

The way you "say" a decimal is how you write it as a fraction.

Change to a fraction: 0.8

"8 <u>ten</u>ths"

$0.8 = \frac{8}{10} \xrightarrow{\text{Reduce}} \boxed{\frac{4}{5}}$

Change to a decimal: 0.37

"37 <u>hundred</u>ths"

$0.37 = \boxed{\frac{37}{100}}$

Changing Decimals to Percents

To change a decimal to a percent, move the decimal point 2 places to the right.

(You can also multiply by 100 to change to a percent.)

Change to a percent: 0.36

0.36 ⇨ 0.36 ⇨ 36. ⇨ $\boxed{36\%}$

Change to a percent: 1.9

1.9 ⇨ 1.9**0** ⇨ 190. ⇨ $\boxed{190\%}$

Fill in zero

Changing Percents to Decimals

To change a percent back to a decimal, move the decimal point 2 places to the left.

(You can also divide by 100 to change to a decimal.)

Change to a decimal: 72%

$\boxed{0.72}$ ⇦ .72 ⇦ 72. ⇦ 72.%

Change to a decimal: 8%

$\boxed{0.08}$ ⇦ .08 ⇦ **0**8. ⇦ 8.%

Fill in zero

Change to a decimal: 135%

$\boxed{1.35}$ ⇦ 135. ⇦ 135.%

Change to a decimal: 4.5%

$\boxed{0.045}$ ⇦ **0**4.5 ⇦ 4.5%

Fill in zero

Changing Percents to Fractions

To change a percent to a fraction, write it as a fraction with a denominator of 100.

Change to a fraction: 44%

$$44\% = \frac{44}{100} \xrightarrow{\text{Reduce}} \boxed{\frac{11}{25}}$$

Change to a fraction: 160%

$$160\% = \frac{160}{100} \xrightarrow{\text{Reduce}} \boxed{\frac{8}{5} \text{ or } 1\frac{3}{5}}$$

Changing Fractions to Percents

To change a fraction to a percent, first change the fraction to a decimal.
Then change the decimal to a percent.

Fraction ⇨ Decimal ⇨ Percent

(You can also multiply by 100 to change to a percent.)

Change to a percent: 3/4

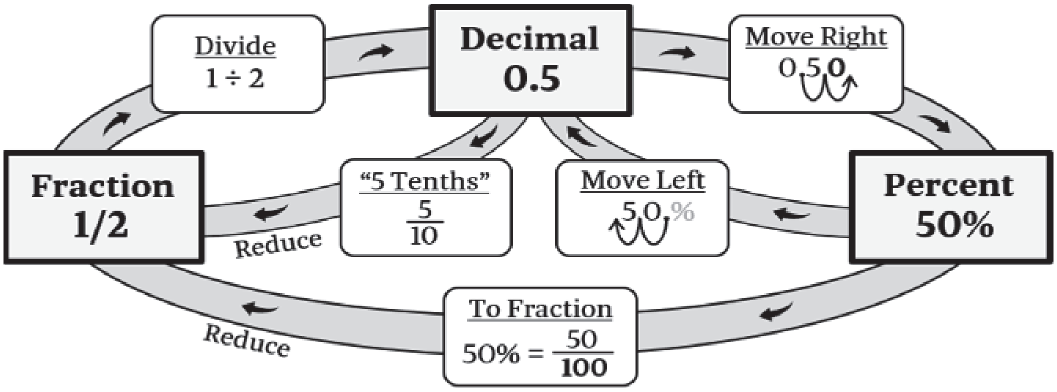
$$\frac{3}{4} \xrightarrow{3 \div 4} 0.75 \Longrightarrow \boxed{75\%}$$

Change to a percent: 1/8

$$\frac{1}{8} \xrightarrow{1 \div 8} 0.125 \Longrightarrow \boxed{12.5\%}$$

Review of Conversions

The flowchart below will help you when changing between fractions, decimals and percents.



Solving Basic Percent Problems

Once again, percents and fractions are just different ways of showing parts of a whole.

For basic percent problems, the easiest way to solve them
is to first write the answer as a fraction, and then change the answer to a percent.

A bag of marbles contains 4 red, 5 blue, 3 green and 8 purple marbles.
What percent of the marbles are green?

First find the fraction of the marbles that are green. Then change the answer to a percent.

3 marbles are green out of 20 total marbles.

$$\frac{\text{3 Green}}{\text{20 Total Marbles}} = \frac{3}{20}$$

Change the fraction to a percent:

$$\frac{3}{20} \Longrightarrow \boxed{15\%}$$

15% of the marbles are green.

A six-sided die weighs 4 grams and a nickel weighs 5 grams.

1. The mass of a die is what percent of the mass of a nickel?

2. The mass of a nickel is what percent of the mass of a die?

Sometimes you may have to write one item as a percentage of another item.

A is what percent of B? $\xrightarrow{\text{SAME AS}}$ Find $\frac{A}{B}$

Write the answers as fractions and change the answers to a percent:

1. A Die is what percent of a Nickel? ⇨ $\frac{4 \text{ Grams}}{5 \text{ Grams}} = \frac{4}{5}$ ⇨ $\boxed{80\%}$ The mass of a die is 80% of the mass of a nickel.

2. A Nickel is what percent of a Die? ⇨ $\frac{5 \text{ Grams}}{4 \text{ Grams}} = \frac{5}{4}$ ⇨ $\boxed{125\%}$ The mass of a nickel is 125% of the mass of a die.

Finding Percents of Amounts

Finding percentages of amounts is another concept that shows up often on ACT tests.

To find the percentage of any amount, *multiply* the percent (as a decimal) by the amount.

There are 18 apples in a basket.
50% of the apples are red.
How many red apples are in the basket?

Find 50% of 18 apples:

50% of 18 ⇨ $0.50 \cdot 18 = \boxed{9 \text{ apples}}$

9 apples are red.

Forty people signed up for a raffle.
5% of the people will win a prize.
How many people won a prize in the raffle?

Find 5% of 40 people:

5% of 40 ⇨ $0.05 \cdot 40 = \boxed{2 \text{ people}}$

2 people won prizes.

Percent Increases, Percent Decreases and Percent Change

Percent Increases and Decreases

In the real world, most people see percent increases and decreases in the form of pay raises, sales tax, store sales and price markups.

They also show up on the ACT, and there are usually two common ways to solve them.

A lamp that originally costs $60 is on sale for 25% off.
What is the sales price of the lamp?

We will solve the problem two different ways.

FIRST WAY:

1. Find the money that was taken off the price.

Find 25% of the cost of the lamp.

25% of \$60
$= 0.25 \cdot \$60 = \$15$

$15 was taken off the price.

2. Subtract to get the sales price.

$\$60 - \$15 = \boxed{\$45}$

The lamp is now $45.

SECOND WAY:

1. Find the percent that is left over after the sale.

The original price is 100% of the price.

$\underset{\text{Old Price}}{100\%} - \underset{\text{Sale}}{25\%} = 75\%$

The new price is 75% of the old price.

2. Work out the sales price.

75% of \$60 ⇨ $0.75 \cdot \$60 = \boxed{\$45}$

The lamp is now $45.

A woman that makes $12 an hour gets an 8% pay raise.
What is the woman's new hourly pay?

Once again, we will solve the problem two different ways.

FIRST WAY:

1. Find the money that was added to the pay.

Find 8% of the cost of her pay.

$$8\% \text{ of } \$12 = 0.08 \bullet \$12 = \$0.96$$

$0.96 was added to her pay.

2. Add to get her new pay rate.

$$\$12 + \$0.96 = \boxed{\$12.96}$$

Her pay is now $12.96 per hour.

SECOND WAY:

1. Find the percent that is left after the raise.

The original pay is 100% of the pay.

$$\underset{\text{Old Pay}}{100\%} + \underset{\text{Raise}}{8\%} = 108\%$$

Her new pay is 108% of her old pay.

2. Work out the new pay rate.

$$108\% \text{ of } \$12 \Rightarrow 1.08 \bullet \$12 = \boxed{\$12.96}$$

The pay is now $12.96 per hour.

Solving Percent Problems by Making Up Numbers

Sometimes making up your own numbers can help you solve problems that involve percent increases and decreases.

100 is the best starting number to use with percentages.

The dollar amounts are equal to the percentages of 100.

$$90\% \text{ of } \$100 = \boxed{\$90}$$

$$125\% \text{ of } \$100 = \boxed{\$125}$$

The amount you go up or down from 100 is equal to the exact percent it went up or down.

Old Price = $100
New Price = $90

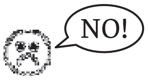
The decrease is 10%.
The new price is 90% of the old price.

Old Price = $100
New Price = $125

The increase is 25%.
The new price is 125% of the old price.

A bike went on sale for 30% off. Then the sales price was marked up by 20%.
The new price of the bike is what percent of the original price of the bike?

This is the wrong answer: ⇨ $\underset{\text{Old Price}}{100\%} - \underset{\text{Sale}}{30\%} + \underset{\text{Markup}}{20\%} = 90\%$ of the old price



Since an original price is not given, we will make up a price to solve the problem.

We will let the original cost of the bike be $100.

First find the sales price.

There was a 30% discount.

30% of $100 = $30

Sales Price
= $100 − $30 = <u>$70</u>

Now find the marked up price.

The sales price was marked up 20%.

20% of $70 = $14

New Price
$= \$70 + \$14 = \boxed{\$84}$

Old Price = $100
New Price = $84

The new price is 84% of the original price.

For completeness, we will also solve the problem algebraically.
Let X be the original cost of the bike.

<u>First find the sales price.</u>

There was a 30% discount.

100% – 30% = 70%

The sales price is 70%
of the original price.

Sales Price = 0.70X

<u>Now find the marked up price.</u>

The sales price was marked up 20%.

100% + 20% = 120%

The new price is 120%
of the sales price.

New Price = 1.20(0.70X) = <u>0.84X</u>

Old Price = X
New Price = 0.84X

The new price is
84% of the original price.

Percent Change

There are times when you may need to calculate how two quantities change percentage-wise from one point in time to the next.

Below is the formula for finding the percent change from an old value to a new value:

$$\frac{\text{New} - \text{Old}}{\text{Old}} \qquad \boxed{\text{Percent Change} = \frac{\text{New Value} - \text{Old Value}}{\text{Old Value}} \text{ x } 100} \qquad \frac{\text{New} - \text{Old}}{\text{Old}}$$

If there's a percent increase, the answer will be positive.
If there's a percent decrease, the answer will be negative.

A bucket contains water that is 16 inches high.
A boy later fills the bucket to 20 inches high.
By what percent did the water's height increase?

<u>Old Value:</u> 16 Inches

<u>New Value:</u> 20 Inches

$$\text{Percent Change} = \frac{20 - 16}{16} \times 100 = 0.25 \times 100$$
$$= \boxed{25\%}$$

There was a 25% increase in height.

The original price of a coat was $75.
A store has a sale and now sells the coat for $30.
What was the percent discount of the coat?

<u>Old Value:</u> $75

<u>New Value:</u> $30

$$\text{Percent Change} = \frac{30 - 75}{75} \times 100 = -0.6 \times 100$$
$$= \boxed{-60\%}$$

The coat was discounted by 60%.

Solving Percent Problems By Translation

Some percent problems can be solved by translating words into simple equations.

Below are examples of basic percent problems that can be solved with an equation:

15% of what number is 9?

18 is 24% of what number?

What is 140% of 85?

When you see the words below in a percent problem, you change them to the symbols below:

IS	⇒	EQUALS =
What Number, What Value What	⇒	MISSING VALUE "X"
OF	⇒	MULTIPLY •

You must change any percents to decimals when solving the equations.

What is 140% of 85?

Let 'X' be the answer to the problem. Change the words to an equation:

$$\underset{X}{\underset{\downarrow}{\text{What}}}\ \underset{=}{\underset{\downarrow}{\text{is}}}\ \underset{1.40}{\underset{\downarrow}{140\%}}\ \underset{\bullet}{\underset{\downarrow}{\text{of}}}\ \underset{85}{\underset{\downarrow}{85?}}$$

Solve the equation:

$$X = 1.40 \bullet 85$$

$$X = \boxed{119}$$

140% of 85 is $\underline{119}$.

15% of what number is 9?

Change the words to an equation:

$$\underset{0.15}{\underset{\downarrow}{15\%}}\ \underset{\bullet}{\underset{\downarrow}{\text{of}}}\ \underbrace{\text{what number}}_{X}\ \underset{=}{\underset{\downarrow}{\text{is}}}\ \underset{9}{\underset{\downarrow}{9?}}$$

Solve the equation:

$$0.15X = 9 \Rightarrow X = \boxed{60}$$

15% of $\underline{60}$ is 9.

18 is 24% of what number?

Change the words to an equation:

$$\underset{18}{\underset{\downarrow}{18}}\ \underset{=}{\underset{\downarrow}{\text{is}}}\ \underset{0.24}{\underset{\downarrow}{24\%}}\ \underset{\bullet}{\underset{\downarrow}{\text{of}}}\ \underbrace{\text{what number?}}_{X}$$

Solve the equation:

$$18 = 0.24X \Rightarrow X = \boxed{75}$$

18 is 24% of $\underline{75}$.

Using the IS/OF Formula

Another way to translate and solve percent problems is to use the IS/OF formula and cross multiply to find the unknown value.

$$\frac{\text{Part}}{\text{Whole}} = \frac{\text{Part (\%)}}{\text{Whole (100\%)}} \xrightarrow{\text{Changes To}} \boxed{\frac{\text{IS}}{\text{OF}} = \frac{\text{PERCENT}}{100}}$$

There are 3 rules you follow when using the formula:

Rule #1

Percents (30%, 65%, 5.5%, 700%) always go into the "PERCENT" spot.

Rule #2

The number that is closest to the word "is" goes into the "IS" part of the formula.

Rule #3

The number that is closest to the word "of" goes into the "OF" part of the formula.

15% of what number is 9?

$$\frac{9}{X} = \frac{15}{100}$$

$$\boxed{\frac{\text{IS}}{\text{OF}} = \frac{\text{PERCENT}}{100}}$$

What percent of 20 is 7?

$$\frac{7}{20} = \frac{X}{100}$$

Percents go into the "percent" spot even if they're closest to the word "of" or "is".

What is 140% of 85?

Break down the problem:

What is 140% of 85?
↓ ↘
Goes into PERCENT
Goes into OF

$$\frac{\text{IS}}{\text{OF}} = \frac{\text{PERCENT}}{100}$$

$$\frac{X}{85} = \frac{140}{100}$$

The "IS" part is missing.

After cross multiplying:

$$X = \boxed{119}$$

18 is 24% of what number?

Break down the problem:

18 is 24% of what number?
↙ ↓
Goes into IS
Goes into PERCENT

$$\frac{\text{IS}}{\text{OF}} = \frac{\text{PERCENT}}{100}$$

$$\frac{18}{X} = \frac{24}{100}$$

The "OF" part is missing.

After cross multiplying:

$$X = \boxed{75}$$

Practice Problems

1. A video game that was originally $36 is marked up by 25%. A coat that was originally $40 is discounted by 20%. If a person buys both items at their new prices and the sales tax rate is 5%, what is the total cost of the items, including sales tax?

A. $76.85
B. $77.65
C. $78.75
D. $80.85
E. $82.75

2. John and Matt work at the same department store. John makes $10 per hour and Matt makes $12 per hour. John gets a 5% pay raise and Matt gets a 6% pay raise. When the pay raises go into effect, how much more money will Matt earn than John whenever they work an 8–hour shift?

A. $16.00
B. $16.80
C. $16.96
D. $17.76
E. $18.00

3. A particular shirt at a store costs $15. If sales tax is 4% on all items sold, how many $15 shirts would need to be bought in order for the sales tax paid to be $7.20?

A. 10
B. 12
C. 14
D. 16
E. 18

4. Kean has 260 sports trading cards. If 30% of the cards he owns are baseball cards and 45% of the cards he owns are football cards, how many trading cards does he own that are *not* football nor baseball?

A. 35
B. 65
C. 185
D. 195
E. 225

5. A basketball player so far has made 30 out of his 50 free throws attempts for a free throw percentage of 60%. If he misses 4 of his next 10 free throws, what will be his updated free throw percentage?

A. 50%
B. 56.7%
C. 60%
D. 64.3%
E. 66.7%

6. Regina visits a taco truck on the street and buys 6 tacos that cost $1.50 each. On top of that, she gives the cook a 15% tip for the tacos. If Regina pays no sales tax for the tacos, how much change would she get back if she gives the cook a $20 dollar bill?

A. $8.75
B. $9.20
C. $9.65
D. $11.00
E. $12.35

7. A laptop computer in a store is dropping in price. Initially, the laptop was put on sale for 10% off. Later on, the store decided to put the laptop on clearance by taking 50% off the *sales price* of the laptop. The clearance price of the laptop is what percent of its original price?

A. 40%
B. 45%
C. 55%
D. 60%
E. 65%

8. At a clothing store, a dress that originally cost $80 is now on sale for $56. What percent was taken off from the original price of the dress?

A. 30%
B. 35%
C. 40%
D. 45%
E. 50%

9. 5% of 80 is equal to 25% of x. What is x?

A. 16
B. 18
C. 20
D. 24
E. 30

10. In a survey of 600 households, 55% of the households are owned by married couples. Of the households that have married couples, 30% of them also have children that reside there. How many of the households surveyed are occupied by married couples with children?

A. 81
B. 99
C. 198
D. 231
E. 510

11. A hotel on a busy night had 800 of its rooms occupied. The next night, only 762 rooms were occupied. By what percent did the number of occupied rooms drop at the hotel?

A. 3.5%
B. 3.75%
C. 4.5%
D. 4.75%
E. 5%

12. What is the sum of 0.25% of $100, 2.5% of $80, and 250% of $60?

A. $152.25
B. $154.50
C. $170.25
D. $172.50
E. $177.00

13. On a sales tax–free weekend, a mother goes out and buys notebooks for her kids before they go back to school. She goes to a store where every item in the store is 20% off. If the retail price of a notebook at the store is $0.80, what is the maximum number of notebooks she can buy on sale with $15?

A. 15
B. 16
C. 20
D. 23
E. 24

14. If x% of 30 is 6, what is x% of 6?

A. 0.6
B. 1.2
C. 1.8
D. 2.4
E. 3.0

Use the situation below to answer questions 15–17.

Three jars of liquid are sitting on a table. The jars are labeled A, B and C. The jars contain 6 ounces, 8 ounces and 10 ounces of liquid respectively.

15. The volume of liquid in Jar A is what percent of the volume of liquid in Jar B?

A. 60%
B. 75%
C. 80%
D. 133%
E. 167%

16. The volume of liquid in Jar C is approximately what percent of the volume of liquid in Jar A?

A. 60%
B. 75%
C. 80%
D. 133%
E. 167%

17. If the liquid in all three jars are combined and poured into one large container, what percentage of the liquid would have come from Jar A?

A. 25%
B. 33%
C. 42%
D. 58%
E. 75%

18. The price of a stock is fluctuating in price. First, the original stock price rose by 30%. After reports of a questionable economic future, the stock price suddenly drops by 40% from its peak stock price. The new price of the stock is what percent of the original price of the stock?

A. 75%
B. 78%
C. 88%
D. 90%
E. 98%

19. Given that 9% of a number is 18, what is 3/10 of that same number?

A. 30
B. 60
C. 90
D. 120
E. 200

20. An item at an electronics store costs d dollars. Due to demand, the item is marked up by p%.
What is the new price of the item in terms of d and p?

A. $d + \frac{dp}{100}$
B. $d + dp$
C. $d + p$
D. $d + 100dp$
E. $d + \frac{p}{100}$

21. If 12 is 15% of a number, what is 200% of that number?

A. 80
B. 120
C. 160
D. 240
E. 360

22. A lawn chair is on sale for 10% off its retail price. If the retail price of the chair is $60 and sales tax is 2.5%, how much would it cost someone to buy the discounted chair including sales tax?

A. $54.00
B. $54.15
C. $55.35
D. $61.50
E. $67.50

23. The price of a box of popcorn in a movie theater was raised by 20% from January to February and raised by another 30% from February to March. What was the percent increase in the price of popcorn from January to March?

A. 50%
B. 54%
C. 55%
D. 56%
E. 60%

<u>Use the situation below to answer questions 24–27.</u>

Forty math and science books are sitting on a bookshelf in a library. The composition of the books on the shelf are listed in the table.

Subject	Number of Books
Algebra	15
Geometry	10
Chemistry	9
Biology	6

24. What percent of all the books are math books?

A. 25%
B. 37.5%
C. 62.5%
D. 77.5%
E. 85%

25. What percent of all the books are biology books?

A. 6%
B. 15%
C. 22.5%
D. 25%
E. 37.5%

26. The number of chemistry books make up what percent of the science books on the shelf?

A. 22.5%
B. 25%
C. 37.5%
D. 40%
E. 60%

27. The number of geometry books is what percent of the number of math books on the shelf?

A. 22.5%
B. 25%
C. 37.5%
D. 40%
E. 60%

28. 1.35% of what number is 54?

A. 4
B. 40
C. 400
D. 4000
E. 40000

29. A college textbook that was priced at $72 last year is now being sold at a bookstore for $90 this year. By what percent was the textbook marked up?

A. 18%
B. 25%
C. 35%
D. 45%
E. 75%

30. In a nationwide survey, 56% of millennials are not married. What fraction of the millennials surveyed *are* married?

A. 9/20
B. 11/20
C. 11/25
D. 14/25
E. 13/50

31. 3/4 of 48 is equivalent to 25% of what number?

A. 144
B. 156
C. 164
D. 172
E. 196

32. What is $12\frac{1}{2}$% of $1\frac{1}{5}$?

A. 0.145
B. 0.150
C. 0.175
D. 0.185
E. 0.250

Solutions: 1D, 2D, 3B, 4B, 5C, 6C, 7B, 8A, 9A, 10B, 11D, 12A, 13D, 14B, 15B, 16E, 17A, 18B, 19B, 20A, 21C, 22C, 23D, 24C, 25B, 26E, 27D, 28D, 29B, 30C, 31A, 32B

1.6 Money Problems and Amount Matching

In the previous section, money problems that involved percentages were covered. In this section, we will focus on money problems that *do not* involve percentages.

Types of Money Word Problems

Initial Fee + Rate Problems

In these types of problems, there is a fee you must pay upfront. Then you get charged a certain amount for continuing the service.

> An insurance plan charges an initial fee of $50 and requires a recurring payment of $15 per month. What is the total cost to keep the plan for 6 months?

Each month costs $15:

$15 × 6 months = $90

6 months cost $90

Now add the $50 fee:

$$\underbrace{\$90}_{\text{6 months}} + \underbrace{\$50}_{\text{Fee}} = \boxed{\$140}$$

The total cost is $140 for 6 months of insurance.

Sometimes you may have to calculate how long you can keep a service if you put down a certain amount of money at the beginning.

> An insurance plan charges an initial fee of $50 and requires a recurring payment of $15 per month. A man has $170 to pay for his fee and insurance plan. How many months can he pay for?

There are several ways to solve the problem.

FIRST WAY:

First, subtract out the fee:

$170 – $50 = $120

$120 is left over.

Now divide to see how many months you can buy with $120:

$$\$120 \div \underbrace{\$15}_{\text{Monthly Fee}} = \boxed{\text{8 months}}$$

The man can buy 8 months of insurance.

SECOND WAY:

We can set up an equation to find out how many months the man can pay for.

Let n = Number of Months:

$$50 + 15n = 170$$

$$\text{Initial Fee} + \left(\text{Monthly Fee} \times \text{Number of Months}\right) = \text{Money to Spend}$$

Solve the equation:

$50 + 15n = 170$ Subtract 50 on both sides

$15n = 120$ Divide 15 on both sides

$\boxed{n = 8}$

The man can buy 8 months of insurance.

Another situation where an initial fee plus a rate applies is when mass producing products.

Usually, an initial cost is put down to buy the factory or machinery to make the products. Then each product costs a certain amount to produce.

Kala is selling homemade soaps. She buys a soap-making machine for $800. It costs about $1.75 in materials to make each bar of soap.

What expression can be used to model the total cost of making *b* bars of soap?

Total Cost = $800 + ($1.75 × Number of Soaps) ⇨ Cost = 800 + 1.75b OR Cost = 1.75b + 800

Amount Matching Problems

Sometimes you may be given two different plans and you will need to figure out when the plans will come out costing the same amount of money.

To get a gym membership at Tight Abs Fitness Club, you must pay an application fee of $15 and a monthly charge of $25 a month. To get a membership at Big Biceps Fitness Club, you must pay an application fee of $35 and a monthly charge of $20 a month.

After how many months will the amount of money paid under both plans will be equal?

We will also solve the problem two different ways.

FIRST WAY:

We can calculate the cost from month to month until we see that the costs are equal.

	1st Month		2nd Month		3rd Month		4th Month
Tight Abs Fitness	$15 + $25 = $40 Fee Month	+$25	$65	+$25	$90	+$25	$115
Big Biceps Fitness	$35 + $20 = $55 Fee Month	+$20	$75	+$20	$95	+$20	$115

The costs will be the same after 4 months.

SECOND WAY:

We can set up an equation to find out when the membership plans will be equal in cost.

Let n = Number of Months:

Tight Abs Fitness: $15 + ($25 × Number of Months)

Big Biceps Fitness: $35 + ($20 × Number of Months)

$$15 + 25n = 35 + 20n$$

Solve the equation:

$$15 + 25n = 35 + 20n \quad \text{Subtract 20n}$$
$$15 + 5n = 35 \quad \text{Subtract 15}$$
$$5n = 20 \quad \text{Divide 5}$$
$$\boxed{n = 4}$$

The costs will be equal after 4 months.

Amount matching is also used in situations other than money.

A bucket that contains 2 gallons of water is getting filled at a rate of 0.25 gallons per minute. Another bucket that contains 11 gallons of water is losing water at a rate of 0.5 gallons per minute. After how many minutes will the amount of water be the same in both buckets?

The first way to solve the problem is to calculate the water level every minute for each bucket until their water levels are the same.

The second way to solve the problem is to set up an equation.

Let m = Number of Minutes:

First Bucket:

$2 + \left(0.25 \times \begin{matrix}\text{Number}\\ \text{of Minutes}\end{matrix}\right)$

Second Bucket:

$11 - \left(0.5 \times \begin{matrix}\text{Number}\\ \text{of Minutes}\end{matrix}\right)$

$$2 + 0.25m = 11 - 0.5m$$

Solve the equation:

$2 + 0.25m = 11 - 0.5m$ Add 0.5m

$2 + 0.75m = 11$ Subtract 2

$0.75m = 9$ Divide 0.75

$\boxed{m = 12}$ They are equal after 12 minutes.

Work + Overtime Problems

Overtime is an increase in pay that people get for working above 40 hours a week.

In an overtime pay system, employers pay 1 ½ or 1.5 times a person's hourly wage for each hour worked over 40 hours in a week.

Below is an example of finding the total pay of a person who worked overtime.

Deneka's normal pay rate is $8.30 an hour for all hours worked up to 40 hours a week. For every hour above 40 hours in a week, she gets paid 1½ times her normal pay rate. How much money does Deneka earn for working 48 hours in a single week?

Find the money earned for the first 40 hours:

$8.30 × 40 hours = <u>$332</u>

Calculate the overtime hourly rate:

$8.30 × 1.5 = $12.45 per hour

Overtime = 48 – 40 = 8 extra hours

Find the money earned for 8 hours of overtime:

$12.45 × 8 hours = <u>$99.60</u>

Now add together the money earned for regular hours and overtime hours to get the total pay:

$$\underset{\text{Regular Pay}}{\$332} + \underset{\text{Overtime Pay}}{\$99.60} = \underset{\text{Total Pay}}{\boxed{\$431.60}}$$

Deneka earned $431.60 for the week.

Profit Problems

Revenue is the money you make from selling goods and services.

Profit is the revenue or money you keep after paying for costs and expenses.

$ Profit = Revenue – Costs $

Doug is selling his novels for one day at a book fair. It costs him $50 to rent the table for a day. He sells each book for $12 each. The cost to print each book was $4 each.

What expression can be used to model the profit made in selling *n* novels at the book fair?

There are two ways to look at the problem.

FIRST WAY:

Doug makes a profit of $12 – $4 = $8 on each book.

For 'n' novels, he will make a profit of 8n.
($8 Profit × Novels Sold → 8n)

He also needs to subtract off the $50 table.

$$\boxed{\text{Profit} = 8n - 50}$$

SECOND WAY:

$$\text{Profit} = \underset{(\$12 \times \text{Novels})}{\text{Revenue from Books}} - \underset{(\$4 \times \text{Novels})}{\text{Cost of Making Books}} - \underset{(\$50)}{\text{Cost of Table}}$$

$$= 12n - 4n - 50$$

$$= 8n - 50$$

$$\boxed{\text{Profit} = 8n - 50}$$

Other Types of Money Problems

Sometimes you are asked to calculate the difference in the cost of two situations.

Lisa has a credit card balance of $1200. She started off by making an initial payment of $300 and then made 24 additional payments of $60 each to bring her credit card balance to zero. If you include all the payments, how much money would Lisa had saved had she paid off the credit card bill in one big payment?

We already know it costs $1200 to make a one-time payment.
All we have to do now is calculate the total cost for the 25 payments:

Total amount of money spent by Lisa:

$$\underset{\text{1st Payment}}{\$300} + \underset{\text{Next 24 Payments}}{(\$60 \times 24 \text{ Payments})} = \$1740$$

Find the difference in cost:

$$\$1740 - \$1200 = \boxed{\$540}$$

Lisa would have saved $540.

Sometimes you may need to use one item to find the price or quantity of another item.

A school is selling adult and child tickets to a play. The cost of an adult ticket is $8.75. If four adults and seven children can attend the play for a total of $71.75, what is the cost of a single child ticket?

We know each adult ticket costs $8.75.
Use the adult ticket price to find the cost of each child ticket:

Find the cost of four adult tickets:

$8.75 × 4 Adults = $35

Subtract the adult tickets from $71.75:

$$\underset{\text{Total}}{\$71.75} - \underset{\text{Adults}}{\$35} = \underset{\text{Left Over}}{\$36.75}$$

Now get the cost of a child ticket:

$$\$36.75 \div 7 \text{ Kids} = \boxed{\$5.25}$$

A child ticket costs $5.25.

We also could have used the equation $(4 \bullet 8.75) + 7c = 71.75$ to find the cost of a child ticket.

Practice Problems

1. A library is selling used paperback and hardback books at their bookstore. They are charging \$0.25 for each paperback book and \$0.75 for each hardback book. So far, the library has made \$70 selling used books. If they sold 58 hardback books, how many paperback books did they sell?

A. 106
B. 108
C. 110
D. 112
E. 114

2. A person goes into the deli section of a grocery store and buys 10 chicken wings and 8 chicken fingers. Not including tax, the total cost of the food was \$18.70. If each chicken wing costs 95 cents each and each chicken finger costs d dollars, which equation would give you the cost, in dollars, of a single chicken finger?

A. $10d + 8(95) = 18.70$
B. $10(0.95) + 8d = 18.70$
C. $10(8)(0.95d) = 18.70$
D. $10(95) + 8d = 18.70$
E. $10d + 8(0.95) = 18.70$

3. James works at a retail store and earns \$9.40 per hour. For each hour he works over 40 hours in a week, he gets paid an overtime rate of $1\frac{1}{2}$ times his normal pay rate. What would be John's total pay for a week in which he worked 50 hours?

A. \$470
B. \$503
C. \$510
D. \$517
E. \$705

4. A man is getting one of the parts inside his car replaced by a mechanic. The cost of the part is \$75.50 and the mechanic charges \$55 per hour of labor. If the bill for the repair was \$350.50, how many hours were charged for labor?

A. 3
B. 4
C. 5
D. 6
E. 7

5. Clarissa is buying a piano for \$12,000. She makes an initial payment of \$1,500 and then gets a loan to pay off the remaining balance. For the loan, she must make monthly payments of \$360 a month for 36 months. When you add up the down payment and all her monthly payments, the amount is how much more than the original price of the piano?

A. \$960
B. \$2,100
C. \$2,460
D. \$2,820
E. \$3,180

6. A bakery is selling quantities of cupcakes at different prices. A single cupcake costs \$2.75 each, a half–dozen cupcakes costs \$15.25, and a dozen cupcakes cost \$29.00.

If a person visits the bakery and decides to buy exactly 35 cupcakes, what is the least amount of money it would cost to buy the cupcakes, not including tax?

A. \$87.00
B. \$88.50
C. \$89.75
D. \$90.25
E. \$93.00

7. A woman sells scented soaps and lotions for a beauty supply company. She earns a \$4 commission for each bar of soap she sells and a \$5 commission for each bottle of lotion she sells. This month, she sold b bars of soap and L bottles of lotion. Which expression represents the total commissions earned for the month from selling soap and lotion?

A. $9(L + b)$
B. $4L(5b)$
C. $4(5)(L + b)$
D. $4L + 5b$
E. $4b + 5L$

8. Henry is selling pottery on the side to make more money to pay his bills. It costs him $7,500 to purchase the equipment. It costs $25 to make each pot, so he decides to sell the pots for $60 each. If Henry sells p pots, which of the following must be true?

I. The revenue made for selling p pots is $60p$

II. The combined costs for selling p pots is $25p + 7500$

III. The profit generated from selling p pots since starting the business is $35p - 7500$

A. I only
B. I and II only
C. I and III only
D. II and III only
E. I, II and III

9. Emily decides to start a t-shirt business. She has to pay $3,600 upfront to start the business. If each t-shirt costs $2.50 to make, and she sells each t-shirt for $10 each, how many t-shirts will she have to sell to break even?

A. 460
B. 480
C. 500
D. 520
E. 540

10. A music streaming service requires an initial fee of $15 and a recurring fee of $4 a month. If a person pays $100 in advance for the service, what is the maximum number of months they can use the service?

A. 18
B. 19
C. 20
D. 21
E. 22

11. Two different machines are making toys. Machine A has already produced 750 toys and makes 20 toys per hour. Machine B has already produced 500 toys and produces 25 toys per hour. After how many hours will the total amount of toys produced by both machines will be equal?

A. 40
B. 45
C. 50
D. 55
E. 60

12. At a movie theater, a group of friends bought 3 large bags of popcorn that cost $8.50 each. In addition to the popcorn, the group bought 4 medium drinks of soda. If the group paid a total of $44.90 for popcorn and drinks, what was the cost for a medium soda?

A. $4.10
B. $4.35
C. $4.60
D. $4.85
E. $5.10

13. Jacob and Lucy both have bank accounts. Each day, Jacob withdraws $10 of his cash each day whereas Lucy deposits $25 to her bank account each day. On January 1st, Jacob had $880 in his bank account, and Lucy had $250 in her bank account. After how many days will the money in both bank accounts be the same amount?

A. 16
B. 18
C. 21
D. 25
E. 30

14. A coach purchases x soccer balls and y jerseys for his soccer team at a sports store. The soccer balls cost a total of $180. If the coach spent $720 at the store, which expression gives the combined cost of 1 soccer ball and 1 jersey? (You can disregard sales tax for this problem.)

A. $\frac{180}{x} + \frac{540}{y}$
B. $180x + 540y$
C. $x + y$
D. $540x + 180y$
E. $\frac{540}{x} + \frac{180}{y}$

15. A woman is renting a car. Driving the rental car will cost her $45 per day and $0.40 per mile. If she rents the car for Monday, Tuesday and Wednesday and ends up with a total bill of $145.80, how many miles did she travel with the car?

A. 25
B. 26
C. 27
D. 28
E. 29

Use the situation below to answer questions 16–17.

A taxicab service charges an initial fee of $5 to use the service and $2.70 per mile to drive to the destination.

16. Mary decided to use the service. If the cost of her trip was $23.90, how many miles were driven on the trip?

A. 4
B. 5
C. 6
D. 7
E. 8

17. Bob also decides to use the service. If he needs the taxi to travel 12 miles to get him home, what will be the cost of the trip?

A. $32.40
B. $35.10
C. $37.40
D. $40.10
E. $42.80

18. An organization is having a bake sale to raise money. They are selling cookies for $2.50 each and muffins for $3.25 each. They sold twice as many cookies as muffins. If m represents the number of muffins sold, which expression represents the total amount of money that was earned from selling cookies and muffins at the bake sale?

A. $2.50m$
B. $3.25m$
C. $5.75m$
D. $8.25m$
E. $8.75m$

19. An amusement park is offering special packages to families. A family of 4 can get a one-day pass to the park for a total of $78. A family of 5 can get a one-day pass to the park for a total of $82. How much cheaper does it cost per person to attend the park under the 5-person deal compared to the 4-person deal?

A. $3.10
B. $3.30
C. $3.50
D. $3.70
E. $3.90

20. David is buying a $4,800 treadmill. He makes a down payment of $250 and gets a loan to pay off the remaining balance. For the loan, he must make monthly payments of $175 a month for 36 months. How much extra will David have to pay overall by not paying for the treadmill in one large payment?

A. $1,500
B. $1,750
C. $1,925
D. $2,100
E. $2,275

21. A cell phone plan from Phone Company A requires an activation fee of $65 and a monthly payment of $40 a month. A cell phone plan from Phone Company B has a $15 activation fee and a monthly payment of $50 a month. If both cell plans are started at the same time, after how many months will the total spent from each plan will be equal?

A. 5
B. 6
C. 7
D. 8
E. 9

22. A monthly phone service charges 4 dollars to make an international phone call, plus an additional 50 cents per minute for each minute connected to the call. Which equation below shows the association between m, the amount of minutes connected to the call, and c, the full cost of the international call, in dollars?

A. $c = 0.5 + 4m$
B. $m = 4 + 0.5c$
C. $c = 4 + 50m$
D. $m = 50 + 4c$
E. $c = 4 + 0.5m$

Solutions: 1A, 2B, 3D, 4C, 5C, 6A, 7E, 8E, 9B, 10D, 11C, 12D, 13B, 14A, 15C, 16D, 17C, 18D, 19A, 20B, 21A, 22E

1.7 Ratios, Rates and Proportions

Writing Ratios

A **ratio** is a comparison between two or more objects or things.

There are 3 ways to write a ratio:

$$A : B = \frac{A}{B} = A \text{ to } B$$

What is the ratio of arrows to stars in the picture?

We have 3 arrows to 2 stars.

The ratio of arrows to stars is 3 to 2.

$$3 : 2 = \frac{3}{2} = 3 \text{ to } 2$$

Writing Ratios in the Correct Order

With ratios, the order you write them matters.

On a farm, there are 11 cows and 4 horses.

1. Find the ratio of cows to horses.
2. Find the ratio of horses to cows.

There are 11 cows to 4 horses.

The ratio of cows to horses = 11 to 4

There are 4 horses to 11 cows.

The ratio of horses to cows = 4 to 11

Reducing and Simplifying Ratios

Since ratios can be written as fractions, they can also be reduced to lowest terms.

In a paper bag there are 9 black balls,
8 gray balls and 6 white balls.

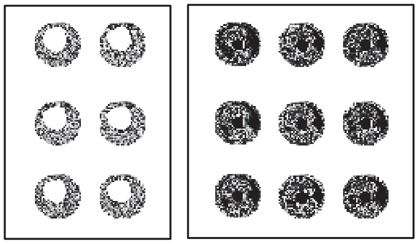
Find the ratio of white balls to black balls.

The ratio of white balls to black balls is 6 : 9

$$6 : 9 = \frac{6}{9} \xrightarrow{\text{Reduce}} \frac{6 \div 3}{9 \div 3} = \boxed{\frac{2}{3}}$$

The ratio can be reduced to 2 : 3

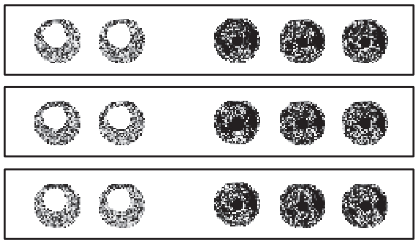
There are 6 white balls for every 9 black balls.



SAME AS

$6 : 9 = 2 : 3$

There are 2 white balls for every 3 black balls.



Three–Part Ratios

We can write ratios as a comparison of three items as well.

What is the ratio of stars to circles to arrows in the picture?

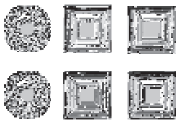
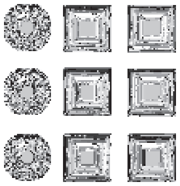
We have 2 stars to 1 circle to 3 arrows.

$\boxed{2:1:3}$ or $\boxed{\text{2 to 1 to 3}}$

Equivalent Ratios and Multiplication

You can make an equivalent ratio by multiplying the entire ratio by the same number.

All the ratios below are equivalent. There is 1 circle for every 2 squares.

$\overset{\times 1}{1} : \overset{\times 1}{2}$ ⇩ $\boxed{1 : 2}$ 	$\overset{\times 2}{1} : \overset{\times 2}{2}$ ⇩ $\boxed{2 : 4}$ 	$\overset{\times 3}{1} : \overset{\times 3}{2}$ ⇩ $\boxed{3 : 6}$ 

"Sum" Ratio Problems

Sometimes the ACT will give you a ratio of two things that add up to some value.

Adam and Mary have ages that are in a ratio of 5 : 7.
If their ages add up to 36, find their ages.

We will solve the problem two different ways.

FIRST WAY:

Set up the problem:

We do not know Adam or Mary's age.

(They are clearly not 5 and 7 years old.)

Use "x" as a multiplier
to keep their ages at a ratio of 5 : 7.

Adam's Age		Mary's Age		
5x	+	7x	=	36

Solve the equation:

$$5x + 7x = 36$$

$$12x = 36$$

⇩

$$\frac{\cancel{12}x}{\cancel{12}} = \frac{36}{12}$$

$$x = 3$$

Plug in x = 3
to get their ages.

Find their ages:

Adam $= 5x = 5 \cdot 3$
= *15 years old*

Mary $= 7x = 7 \cdot 3$
= *21 years old*

They still have a ratio of 5:7.

$15{:}21 = \frac{15 \div 3}{21 \div 3} = \frac{5}{7} = 5{:}7$ ✓

SECOND WAY:

We can multiply the ratio by a number so that their ages add up to 36.

$$\overset{+3}{5} : \overset{+3}{7} = \underset{\text{Adam}}{15} : \underset{\text{Mary}}{21}$$

The ages now add up to 36.
$15 + 21 = 36$

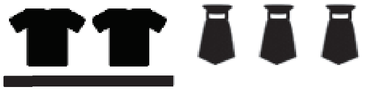
Adam = *15 years old* , Mary = *21 years old*

Combining Ratios

If two ratios have the exact same thing in common,
we can combine the ratios to make a new ratio.

Inside a closet are shirts, ties and hats. The ratio of shirts to ties is 2:3.
The ratio of ties to hats is 3:5. What is the ratio of shirts to hats in the closet?

There are 2 shirts for every 3 ties:



There are 3 ties for every 5 hats:

For every 3 ties,
we will have 2 shirts and 5 hats.

The ratio of shirts to hats is 2 : 5

In a fruit basket, the ratio of apples to pears is 5 to 6.
The ratio of lemons to apples is 14 to 5.
Find the ratio of lemons to pears.

Both ratios use the number of apples.
We can combine the ratios.

For every 5 apples,
we have 14 lemons and 6 pears.

The ratio of lemons to pears is 14 : 6.

Reduce the answer:

$$14 : 6 = \frac{14}{6} \xrightarrow{\text{Reduce}} \frac{14 \div 2}{6 \div 2} = \boxed{\frac{7}{3}}$$

The ratio of lemons to pears is 7 : 3

Let’s look at an example where two ratios have something in common,
but the amounts are not identical in both ratios.

At a picnic, the ratio of boys to girls is 3:4.
The ratio of pets to girls is 2:5.
Find the ratio of boys to pets at the picnic.

The ratios do not have the same number of girls.
We need to make them the same.

Find the LCM to make the girls match:

3 boys to *4 girls*	2 pets to *5 girls*
Multiples of 4: 4, 8, 12, 16, *20*, 24…	Multiples of 5: 5, 10, 15, *20*, 25…

We can multiply and give both ratios 20 girls.

Change each ratio to 20 girls:

$$\overset{\times 5}{3} : \overset{\times 5}{4} = \underset{\text{BOYS}}{15} : \underset{\text{GIRLS}}{20} \qquad \overset{\times 4}{2} : \overset{\times 4}{5} = \underset{\text{PETS}}{8} : \underset{\text{GIRLS}}{20}$$

For every 20 girls,
we have 15 boys and 8 pets.

The ratio of boys to pets is 15 : 8

Rates and Unit Rates

Rates are ratios where the units are always shown.

In many cases, we use the word "per" to show a rate.
"Per" means 'for every' or 'for each'.

A man drove 50 miles every 2 hours.

$$\frac{\text{50 miles}}{\text{2 hours}} = \boxed{\text{50 miles per 2 hours}}$$

It costs $3.99 for each pound of chicken.

$$\frac{\$3.99}{\text{1 pound}} = \boxed{\text{\$3.99 per pound}}$$

The rate of "$3.99 per pound" is called a **unit rate**.

In unit rates, the first unit is related to 1 part of the second unit.

If you want to change a normal rate into a unit rate,
just divide in the exact order you want the rate to be written.

A car travels 80 miles on 4 gallons of gas.
What is the average number
of miles travelled per gallon of gas?

We want miles per gallon.
Divide the amounts in that order:

$\frac{\text{80 miles}}{\text{4 gallons}}$ ⇨ 80 ÷ 4 = 20 ⇨ 20 miles for each gallon

20 miles per gallon

A car travels 80 miles on 4 gallons of gas.
What is the average number
of gallons consumed per mile?

We want gallons per mile.
Divide the amounts in that order:

$\frac{\text{4 gallons}}{\text{80 miles}}$ ⇨ 4 ÷ 80 = 0.05 ⇨ 0.05 gallons for each mile

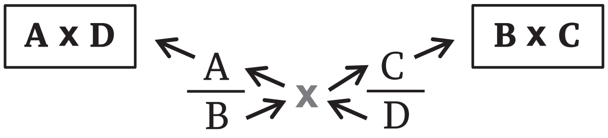
0.05 gallons per mile

Solving Proportions with Cross Multiplication

A **proportion** is an equation that has one fraction on each side.

$\frac{2}{4} = \frac{6}{12}$ Proportions $\frac{4}{5} = \frac{16}{20}$

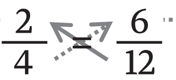
We can cross multiply to show that two fractions are equal.



When two fractions are equal,

A x D = B x C

These fractions are equal. $\frac{2}{4} = \frac{6}{12}$

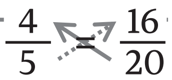


$2 \cdot 12 = 24$

$4 \cdot 6 = 24$

24 = 24

These fractions are equal. $\frac{4}{5} = \frac{16}{20}$



$4 \cdot 20 = 80$

$5 \cdot 16 = 80$

80 = 80

If we know that two fractions are equal, we can cross multiply to find unknown values.

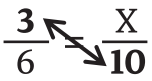
We can solve for X in a proportion with or without using an equation.

Solve Proportions Without an Equation

Find the value of X:

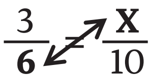
$\frac{3}{6} = \frac{X}{10}$

STEP 1:
Cross multiply the diagonal that does not use X.



$3 \cdot 10 \Rightarrow 30$

STEP 2:
Divide by the number that is cross multiplied with X.

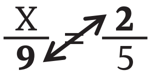


$30 \div 6 \Rightarrow \boxed{5}$

Find the value of X:

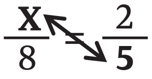
$\frac{X}{9} = \frac{2}{5}$

Cross multiply the diagonal that does not use X.



$9 \cdot 2 \Rightarrow 18$

Divide by the number that is cross multiplied with X.



$18 \div 5 \Rightarrow \boxed{3.6}$

Solve Proportions With an Equation

Find the value of X:

$$\frac{3}{6} = \frac{X}{10}$$

STEP 1:
Cross multiply the diagonals.

$$3 \cdot 10 \leftarrow \frac{3}{6} = \frac{X}{10} \dashrightarrow 6 \cdot X$$

$= 30$ $\qquad$ $= 6X$

$$\boxed{6X = 30}$$

STEP 2:
Solve the equation for X.

Divide by 6:

$$\frac{\cancel{6}X}{\cancel{6}} = \frac{30}{\cancel{6}}$$

$$X = \boxed{5}$$

Find the value of X:

$$\frac{X}{9} = \frac{2}{5}$$

Cross multiply the diagonals.

$$X \cdot 5 \leftarrow \frac{X}{9} = \frac{2}{5} \dashrightarrow 9 \cdot 2$$

$= 5X$ $\qquad$ $= 18$

$$\boxed{5X = 18}$$

Solve the equation for X.

Divide by 5:

$$\frac{\cancel{5}X}{\cancel{5}} = \frac{18}{\cancel{5}}$$

$$X = \boxed{3.6}$$

Cross Multiplying to Solve Word Problems

With some word problems, you are given an old ratio and you need to use them to find a new ratio. Many of these types of problems can solved by cross multiplying.

On a warm day, the temperature rose 15 degrees in 6 hours.
If the temperature keeps going up at the same rate, how long will it take to rise 25 degrees?

When you set up the problem, always write the ratios in the *same order*.

This is OK	This is OK	This is NOT OK
$\frac{15\text{ degrees}}{6\text{ hours}} = \frac{25\text{ degrees}}{X\text{ hours}}$	$\frac{6\text{ hours}}{15\text{ degrees}} = \frac{X\text{ hours}}{25\text{ degrees}}$	$\frac{15\text{ degrees}}{6\text{ hours}} = \frac{X\text{ hours}}{25\text{ degrees}}$
⇩	⇩	⇩
$\frac{15}{6} = \frac{25}{X}$	$\frac{6}{15} = \frac{X}{25}$	$\frac{15}{6} = \frac{X}{25}$
The ratios are in the same order.	The ratios are in the same order.	The ratios are not written in the same order.
$\boxed{\frac{\text{degrees}}{\text{hours}} = \frac{\text{degrees}}{\text{hours}}}$	$\boxed{\frac{\text{hours}}{\text{degrees}} = \frac{\text{hours}}{\text{degrees}}}$	You will not get the correct answer.

Word problems where quantities are going up or going down at the same rate can usually be solved with a proportion.

Key words in rates and ratios word problems:

- "Same Rate" or "Same Ratio"
- "At this Rate" or "At that Rate"
- Increase or Decrease "Proportionally"
- Increase or Decrease "Linearly"

Also, word problems that use recipes,
maps and scale drawings can also be solved by cross multiplying.

For a state map, ¾ inch is equal to 10 miles.
If two cities on the map are 5 ¼ inches apart, how many miles apart are the two cities?

Set up the problem:

Old Ratio: New Ratio:

$$\frac{0.75 \text{ inches}}{10 \text{ miles}} = \frac{5.25 \text{ inches}}{X \text{ miles}}$$

NOTE:
¾ = 0.75
5 ¼ = 5.25

$$\frac{0.75}{10} = \frac{5.25}{X}$$

Cross multiply and solve for X:

$$0.75 \bullet X \leftarrow \frac{0.75}{10} = \frac{5.25}{X} \dashrightarrow 10 \bullet 5.25$$

$= 0.75X$ $= 52.5$

$$0.75X = 52.5$$

$$\boxed{X = 70}$$

The cities are 70 miles apart.

A kid bought five bags of candy for $12.50.
If each bag is the same price, how much does it cost to buy 8 bags of candy?

We will solve the problem two different ways.

FIRST WAY:
Set up a proportion and cross multiply.

SECOND WAY:
Find the cost of 1 bag.
Then multiply by 8 to find the cost of 8 bags.

Old Ratio: New Ratio:

$$\frac{5 \text{ bags}}{\$12.50} = \frac{8 \text{ bags}}{\$\,X}$$

$$\frac{5}{12.50} = \frac{8}{X}$$

$$\boxed{X = 20}$$

Find the cost of 1 bag of candy:

$$\underset{\text{Cost}}{\$12.50} \div \underset{\text{Bags}}{5} = \$2.50 \text{ per bag}$$

Now find the cost of 8 bags of candy:

$$\$2.50 \times 8 = \boxed{\$20}$$

8 bags of candy cost $20.

Earlier in the ratios chapter we covered how to solve "sum" ratio problems.

These types of problems can also be solved with a proportion,
and we show an example on the next page.

At a birthday party, the ratio of men to women is 5 to 3.
If there were 72 adults at the party, how many of the adults were men?

Use the total adults and the number of men to set up the proportion.

5 Men + 3 Women = 8 Adults

For every 8 adults that are at the party, 5 of the adults will be men.

$$\boxed{\frac{\text{men}}{\text{adults}} = \frac{\text{men}}{\text{adults}}}$$

Old Ratio: New Ratio:

$$\frac{5\text{ men}}{8\text{ adults}} = \frac{X\text{ men}}{72\text{ adults}}$$

⇩

$$\frac{5}{8} = \frac{X}{72}$$

$$\boxed{X = 45}$$

There were 45 men at the party.

We will end with two word problems that can be solved with a proportion.

The ratio of cars to trucks in a parking lot is 3 : 2. If there are 21 cars in the parking lot, how many trucks are in the parking lot?

Old Ratio: New Ratio:

$$\frac{3\text{ cars}}{2\text{ trucks}} = \frac{21\text{ cars}}{X\text{ trucks}}$$

⇩

$$\frac{3}{2} = \frac{21}{X}$$

$$\boxed{X = 14}$$

There are 14 trucks in the parking lot.

A bug is moving at a rate of 15 inches per minute. At that rate, how many inches did it travel in 14 seconds?

Old Ratio: New Ratio:

$$\frac{15\text{ inches}}{60\text{ seconds}} = \frac{X\text{ inches}}{14\text{ seconds}}$$

(We changed 1 minute into 60 seconds to get the same units on each side.)

⇩

$$\frac{15}{60} = \frac{X}{14}$$

$$\boxed{X = 3.5}$$

The bug travelled 3.5 inches in 14 seconds.

Measurement and Dimensional Analysis

There are some basic measurement conversions that are good to know before taking the ACT. You are likely familiar with most of the conversions on this list:

- ⇨ 1 Minute = 60 Seconds
- ⇨ 1 Hour = 60 Minutes
- ⇨ 1 Day = 24 Hours
- ⇨ 1 Week = 7 Days
- ⇨ 1 Kilometer = 1,000 Meters
- ⇨ 1 Meter = 100 Centimeters
- ⇨ 1 Foot = 12 Inches
- ⇨ 1 Yard = 3 Feet

Any other conversions will usually be given to you on the test.

Single Unit Conversions

There are generally two rules we follow when changing one unit to another unit:

When going from a larger unit to a smaller unit, you MULTIPLY:	When going from a smaller unit to a larger unit, you DIVIDE:
Change 7 minutes to seconds: (1 minute = 60 seconds)	Change 24 feet to yards: (1 yard = 3 feet)
7 Minutes ⇨ 7 × 60 = 420 ⇨ 420 Seconds	24 Feet ⇨ 24 ÷ 3 = 8 ⇨ 8 Yards

The rules make sense.

When you take something large and make it smaller, it requires cutting it up into more pieces.
When you take smaller pieces and combine them, they form a small number of large pieces.

A bucket contains sand that weighs 5 pounds, 4 ounces.
If 1/3 of the sand is removed, how much sand is left in pounds and ounces?
(1 pound = 16 ounces)

The easiest way to solve the problem is to convert everything to ounces.
After subtracting off a third of the total ounces, we can convert the ounces back to pounds.

Find the total ounces:

5 pounds ⇨ 80 ounces
4 ounces ⇨ + 4 ounces
84 ounces

⇨

Take away 1/3 of the sand:

1/3 of 84 = 28 oz
84 – 28 = 56 ounces left

⇨

Change 56 oz to pounds and oz:

56 ounces ÷ 16
= 3 R 8
= 3 pounds, 8 ounces

30+ Dimensional Analysis 30+

Dimensional analysis is the process of converting a given rate into a different rate.

When converting rates, you multiply and cancel units until the units you want are left over.

Change 120 meters per hour to meters per minute.

$$\frac{120\text{ meters}}{1\text{ hour}} \Rightarrow \frac{120\text{ meters}}{1\ \cancel{\text{hour}}} \times \frac{1\ \cancel{\text{hour}}}{60\text{ minutes}} = \frac{120\text{ meters}}{60\text{ minutes}} = \boxed{2\text{ meters per minute}}$$

There are two basic rules we follow when changing rates:

1. When you multiply by a fraction, the top and bottom must be equal to each other.
2. The fraction you multiply with should cancel out units in your problem.

Notice how each fraction or ratio is equal on the top and bottom.

1 hour = 60 minutes
3 feet = 1 yard
1 kilometer = 1000 meters

$$\frac{1\text{ hour}}{60\text{ minutes}} \qquad \frac{3\text{ feet}}{1\text{ yard}} \qquad \frac{1\text{ kilometer}}{1000\text{ meters}}$$

Units will only cancel if they are diagonal to each other when multiplying.

The hours units will not cancel:

$$\frac{120\text{ meters}}{1\text{ hour}} \times \frac{60\text{ minutes}}{1\text{ hour}} = \frac{7200\text{ m} \bullet \text{min}}{\text{hr} \bullet \text{hr}}$$

Looks confusing ☹

The hours units will cancel:

$$\frac{120\text{ meters}}{1\cancel{\text{ hour}}} \times \frac{1\cancel{\text{ hour}}}{60\text{ minutes}} = \frac{120\text{ m}}{60\text{ min}}$$

Looks simple ☺

Change 14 yards per second to feet per second

We do not need to change the seconds.
We will cancel out yards and replace them with feet.

Write 14 yards per sec. as a fraction:

Cancel yards:

$14 \bullet 3 = 42$

$$\frac{14\text{ yards}}{1\text{ second}} \Rightarrow \frac{14\cancel{\text{ yards}}}{1\text{ second}} \times \frac{3\text{ feet}}{1\cancel{\text{ yard}}} = \frac{42\text{ feet}}{1\text{ second}} = \boxed{42\text{ feet per second}}$$

Change 1,320 feet per second to miles per hour
(1 mile = 5,280 feet)

We need to cancel feet and change it to miles.
We need to cancel seconds and change it to hours.

Write 1320 ft per sec. as a fraction:

Cancel seconds:

Cancel minutes:

Cancel feet:

Multiply left to right:

$$\frac{1320\text{ ft}}{1\text{ sec}} \Rightarrow \frac{1320\cancel{\text{ ft}}}{1\cancel{\text{ sec}}} \times \frac{60\cancel{\text{ sec}}}{1\cancel{\text{ min}}} \times \frac{60\cancel{\text{ min}}}{1\text{ hr}} \times \frac{1\text{ mile}}{5280\cancel{\text{ ft}}} = \frac{4{,}752{,}000\text{ miles}}{5280\text{ hours}}$$

$$= \boxed{900\text{ miles per hour}}$$

30+ Direct and Inverse Variation 30+

Direct and inverse variation refer to how variables and quantities change in relation to each other.

In a **direct variation**, quantities go up and down in the same direction.

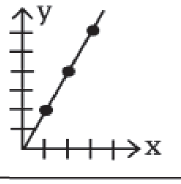
Direct Variation:

"y" varies directly with "x"

$y = kx$

X goes ⇧
Y goes ⇧

X goes ⇩
Y goes ⇩



x	y
1	2
2	4
3	6

$y = 2x$

Below are examples of direct variation and their equations:

Paycheck size varies directly with *hours worked*. ⇨ $P = kH$ (P = paycheck, H = hours)

Circle area varies directly with the *square of its radius*. ⇨ $A = kr^2$ (A = area, r = radius)

In an **inverse variation**, quantities go up and down in opposite directions.

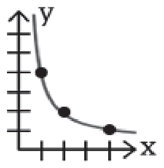
Inverse Variation:

"y" varies inversely with "x"

$y = \frac{k}{x}$

X goes ⇧
Y goes ⇩

X goes ⇩
Y goes ⇧



x	y
1	4
2	2
4	1

$y = \frac{4}{x}$

Below are examples of inverse variation and their equations:

GPA varies inversely with *hours partying.* ⇨ $G = \frac{k}{H}$ (G = GPA, H = hours)

Car value varies inversely with *number of accidents*. ⇨ $V = \frac{k}{A}$ (V = car value, A = accidents)

With both types of variation, *k* is called the **constant of variation**.

The constant of variation (or proportionality) is the ratio of the two variables in a relationship.

X–Y tables with a direct variation:

$y = kx \xrightarrow{\text{Divide by x}} \boxed{\frac{y}{x} = k}$

If a table has a direct variation,
y/x = k for all (x,y) pairs:

$y = 2x$

x	y	k = 2
1	2	⇨ 2/1 = 2
2	4	⇨ 4/2 = 2
3	6	⇨ 6/3 = 2

$y = 5x$

x	y	k = 5
6	30	⇨ 30/6 = 5
8	40	⇨ 40/8 = 5
9	45	⇨ 45/9 = 5

X–Y tables with an inverse variation:

$y = \frac{k}{x} \xrightarrow{\text{Multiply by x}} \boxed{xy = k}$

If a table has an inverse variation,
xy = k for all (x,y) pairs:

$y = \frac{4}{x}$

x	y	k = 4
1	4	⇨ 1 • 4 = 4
2	2	⇨ 2 • 2 = 4
4	1	⇨ 4 • 1 = 4

$y = \frac{12}{x}$

x	y	k = 12
1	12	⇨ 1 • 12 = 12
2	6	⇨ 2 • 6 = 12
3	4	⇨ 3 • 4 = 12

You can also have situations that involve both a direct and inverse variation at the same time.

M varies directly with P ⇨ $M = kP$

M varies directly with Q ⇨ $M = kQ$

M varies inversely with R ⇨ $M = \frac{k}{R}$

⇨ M varies directly with P, directly with Q, and inversely with R:

$M = \frac{kPQ}{R}$

Solving Direct & Inverse Variation Word Problems

Sometimes you will have to solve word problems that involve variation.

<u>These are the basic steps for solving variation word problems:</u>

1. Set up the variation equation.
2. Use the first set of information to find *k*.
3. Use *k* and the second set of info to finish solving the problem.

If it is true that *x* varies inversely with *y*, and *x* = 18 when *y* = 4,
what is *x* when *y* = 3 ?

First, get the equation:
x varies inversely with *y*

$$x = \frac{k}{y}$$

⇨

Use the first set of info to find k:
x = 18, y = 4

$$18 = \frac{k}{4} \Rightarrow 72 = k$$

The formula is $x = \frac{72}{y}$

⇨

Use k and the second set of info:
x = ?, y = 3

$$x = \frac{72}{y} \Rightarrow x = \frac{72}{3} \Rightarrow \boxed{x = 24}$$

When y is 3, x = 24.

The kinetic energy of an object varies directly with the square of its velocity.
If the kinetic energy of the object is 900 joules when its velocity is 10 meters per second,
what is the kinetic energy of the object when it moves at 12 meters per second?

This problem is not as hard as it looks. Let's break it down.

Kinetic energy varies directly with the *square of the velocity*. — $E = kV^2$ (E = energy, V = velocity)

Use the first set of info to find k:
E = 900, V = 10

$$\overset{900}{\downarrow}\ \ \overset{10}{\downarrow}$$
$$E = kV^2 \Rightarrow 900 = k(10)^2$$
$$900 = 100k$$
$$9 = k$$

The formula is $E = 9V^2$

⇨

Now use k and the second set of info:
E = ?, V = 12

$$\overset{12}{\downarrow}$$
$$E = 9V^2 \Rightarrow E = 9(12)^2$$
$$E = 9(144)$$
$$\boxed{E = 1296}$$

The kinetic energy is 1296 joules.

Practice Problems

1. It costs $21 to play 4 games at a bowling alley. How much would it cost to play 11 games at the bowling alley?

A. $47.25
B. $52.50
C. $57.75
D. $63.00
E. $68.25

2. From 1:00pm to 8:00pm, a person sold $3,150 worth of merchandise at a convention. How much money on average did the person make per minute at the convention that day?

A. $7.50
B. $8.10
C. $8.75
D. $9.55
E. $10.50

3. A rack is used for storing ties and belts. The ratio of ties to belts is 9:4. If there are 36 belts in on the rack, how many ties are on the rack?

A. 9
B. 48
C. 54
D. 72
E. 81

4. In a 12–ounce bottle of grape juice, there are 42 grams of sugar. How many grams of sugar should we expect to find in a 16–ounce bottle of grape juice?

A. 35
B. 49
C. 56
D. 63
E. 70

5. Three bugs are moving on a table. In a one-hour period, the first bug moved 2 feet, 9 inches, the second bug moved 3 feet, 8 inches, and the third bug moved 1 foot, 11 inches. What was the combined distance covered by all three bugs?

A. 7 feet, 6 inches
B. 8 feet, 4 inches
C. 8 feet, 6 inches
D. 8 feet, 8 inches
E. 9 feet, 2 inches

6. A 3-foot string of yarn is separated into two pieces. If the ratio of the lengths of the two pieces is 7 to 11, what is the length in *inches* of the larger piece of yarn?

A. 11
B. 14
C. 21
D. 22
E. 33

7. Monica is carrying a purse with only quarters and dimes. There are 3 quarters to every 4 nickels in the purse. If there are 42 coins in her purse, what is the monetary value of Monica's quarters?

A. $4.50
B. $5.25
C. $6.00
D. $7.00
E. $7.50

8. There are forks, spoons and knives in a kitchen. The ratio of forks to spoons is 7:4. The ratio of spoons to knives is 6:7. What is the ratio of forks to knives in the kitchen?

A. 1:1
B. 2:3
C. 3:2
D. 3:4
E. 4:3

9. John spent 75% of his paycheck this month on rent. What is the ratio of the dollar amount that *was not* spent on rent, to the dollar amount of his paycheck that *was* spent on rent?

A. 1:3
B. 1:4
C. 3:1
D. 3:4
E. 4:1

10. In a small city, 3 out of every 100 people are vegetarian. If the city has 54 vegetarians, how many people overall live in the city?

A. 1,600
B. 1,800
C. 2,000
D. 2,200
E. 2,400

11. Two stores are selling packs of the same type of pen. Store A is selling packs of 12 of the pen for $5.76. Store B is selling packs of 18 of the pen for $7.56. What is the ratio of the cost per pen at Store A to the cost per pen at Store B?

A. 2 to 3
B. 3 to 2
C. 3 to 4
D. 7 to 8
E. 8 to 7

12. Janet swims on average 20 hours every week. At this current rate, how many hours (to the nearest tenth) can we expect her to swim in 5 days?

A. 8.6
B. 11.4
C. 14.3
D. 17.1
E. 28.0

13. Emma and Jack both lost weight. Emma lost 34 pounds in 4 months. Jack lost 22 pounds in 3 months. If these rates of weight loss were maintained for an entire 1–year period, what would be the combined weight lost in pounds by both individuals?

A. 168
B. 190
C. 202
D. 212
E. 224

14. A garden has tulips, roses and daisies. There are 8 tulips for every 3 roses. There are 5 daisies to every 2 roses. What is the ratio of daisies to tulips in the garden?

A. 15:16
B. 16:15
C. 24:25
D. 25:24
E. 25:26

15. In a diagram, 1/4 of a centimeter is equivalent to 12 meters. If two objects are 5/8 of a centimeter apart in the diagram, what is the actual distance between the objects in meters?

A. 20
B. 25
C. 30
D. 35
E. 40

16. A particular species of frog can cover approximately 22 inches in 4 jumps. At this rate, about how many inches can the frog cover in 9 jumps?

A. 44
B. 49.5
C. 55
D. 60.5
E. 66

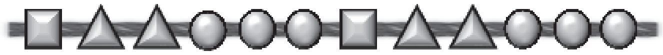
17. What is 1/4 of 5 feet, 4 inches?

A. 1 foot, 2 inches
B. 1 foot, 3 inches
C. 1 foot, 4 inches
D. 1 foot, 6 inches
E. 1 foot, 8 inches

18. Dylan went to class to take a final exam. The overall time he spent taking the exam and checking his work was 2 hours and 6 minutes. For every minute he spent checking his work, eight minutes were spent completing the exam. How much time was spent completing the exam?

A. 1 hour, 28 minutes
B. 1 hour, 34 minutes
C. 1 hour, 40 minutes
D. 1 hour, 46 minutes
E. 1 hour, 52 minutes

19. A child makes a necklace that uses squares, triangles and circles in a ratio of 1:2:3. Below is a portion of the necklace. If there are a total of 24 triangles on the necklace, how many circles are on the necklace?



A. 12
B. 24
C. 36
D. 48
E. 72

20. An insect travels 2 meters in 5 seconds. At this speed, how many seconds did it take for the insect to travel (2 – m) meters?

A. $5 - \frac{5m}{2}$

B. $5 - \frac{2m}{5}$

C. $10 - 5m$

D. $10 - \frac{2m}{5}$

E. $10 - \frac{5m}{2}$

21. There are markers, pencils are erasers in a drawer. The ratio of markers to pencils to erasers is 2 to 4 to 3. If there are 72 items in the drawer, how many of the items are pencils?

A. 16
B. 24
C. 32
D. 40
E. 48

22. The scale of a map is 1 inch:18 miles. If the distance between two cities is 45 miles, how many inches apart are the cities on the map?

A. 2.5
B. 2.75
C. 3
D. 3.25
E. 3.5

23. 50% of x is equal to y% of 75. What is the ratio of x to y?

A. 2 to 3
B. 2 to 5
C. 3 to 2
D. 3 to 5
E. 5 to 3

24. A philanthropic company is running a "Laps to Fight Hunger" event. For every lap a person runs around the track, $5 will be donated to charity. A long-distance runner ran 40 laps at an average pace of 2 minutes per lap. How much charity money did the runner raise per minute of running?

A. $1.50
B. $1.75
C. $2.25
D. $2.50
E. $2.75

25. A recipe for a dipping sauce requires 15 fluid ounces of olive oil, 3/4 teaspoon of oregano and 5 tablespoons of parmesan cheese. If the ingredients keep the same ratio and the recipe is adjusted to use 10 fluid ounces of olive oil, how many teaspoons of oregano would be needed?

A. 1/4
B. 1/8
C. 3/8
D. 1/2
E. 5/8

26. The ratio 1.8 to 0.75 is equivalent to which ratio below?

A. 5:12
B. 6:25
C. 8: 5
D. 12:5
E. 25:6

27. A bucket contains a liquid solution that weighs 3 pounds, 12 ounces. If 90% of the liquid solution is water, what is the weight of the water in the bucket?

A. 3 pounds, 2 ounces
B. 3 pounds, 3 ounces
C. 3 pounds, 6 ounces
D. 3 pounds, 9 ounces
E. 3 pounds, 10 ounces

28. A particular dessert requires 3/8 cup of sugar and 1 1/2 tablespoons of vanilla extract. If I wanted to make more servings of the dessert by increasing all of the ingredients proportionally, how many tablespoons of vanilla extract would I need with 3 1/2 cups of sugar?

A. 13
B. 14
C. 15
D. 16
E. 17

29. A container contains a mixture of salt and sugar that is in a ratio of 1 1/2 liters to 2 1/3 liters. This ratio of salt to sugar is equivalent to:

A. 2 to 7
B. 7 to 2
C. 7 to 9
D. 9 to 14
E. 14 to 9

Practice Problems (30+)

30. A particular insect walks 27 feet per day. What is the insect's walking speed in yards per week?

A. 51
B. 54
C. 57
D. 60
E. 63

NOTE:
(1 yard = 3 feet)

31. If an object is moving at 4 meters per minute, what is its speed in centimeters per hour?

A. 2,400
B. 3,000
C. 15,000
D. 24,000
E. 30,000

NOTE:
(1 meter = 100 centimeters)

32. Which answer choice in meters per second is equivalent to 36 kilometers per hour?

NOTE:
(1 kilometer = 1,000 meters)

A. 10
B. 100
C. 1,000
D. 10,000
E. 100,000

33. Given the values in the table, what form of variation is displayed, and what is the constant of variation?

x	y
1	12
2	6
3	4

A. direct, k = 3
B. inverse, k = 3
C. direct, k = 12
D. inverse, k = 12
E. direct, k = 4/3

34. Based on the values in the table, what type of variation is shown, and what is the constant of proportionality?

x	y
3/4	3
3/2	6
5/2	10

A. direct, k = 4
B. inverse, k = 4
C. direct, k = 9
D. inverse, k = 9
E. direct, k = 25

35. Based on the values in the table, what type of variation is shown, and what is the constant of variation?

x	y
1.2	1.5
0.75	2.4
0.45	4.0

A. direct, k = 1.8
B. inverse, k = 1.8
C. direct, k = 3.2
D. inverse, k = 3.2
E. direct, k = 1.25

36. The length of a test, L, in minutes, varies directly with p, the number of problems on the test. A teacher put 30 problems on a test and gave students 45 minutes to finish the test. What would be the length of a test, in minutes, that contains 50 problems?

A. 60
B. 65
C. 75
D. 80
E. 95

37. It is known that h varies directly with v. If $h = 72$ when $v = 4$, what is the value of h when $v = 7$?

A. 115
B. 120
C. 126
D. 132
E. 138

38. Given that its true that w varies inversely with c, if w takes on a value of 9 when $c = 20$, what value does w take on when $c = 15$?

A. 10
B. 12
C. 15
D. 16
E. 18

39. Let H, A, B and C be variables that only take on positive real number values. The variable H varies directly with C, inversely with the square root of B and directly with A. If k is the constant of variation, which equation displays this relationship?

A. $H = \frac{AC}{k\sqrt{B}}$

B. $H = \frac{kA}{C\sqrt{B}}$

C. $H = \frac{k\sqrt{B}}{AC}$

D. $H = \frac{kC}{A\sqrt{B}}$

E. $H = \frac{kAC}{\sqrt{B}}$

40. The pressure applied to a surface varies inversely with the total area of the surface. When the area of a surface was 6 square meters, the pressure exerted was 24 Pascals. What pressure reading in Pascals would we expect if the area of the surface was increased to 9 square meters?

A. 12
B. 16
C. 18
D. 36
E. 48

Solutions: 1C, 2A, 3E, 4C, 5B, 6D, 7A, 8C, 9A, 10B, 11E, 12C, 13B, 14A, 15C, 16B, 17C, 18E, 19C, 20A, 21C, 22A, 23C, 24D, 25D, 26D, 27C, 28B, 29D, 30E, 31D, 32A, 33D, 34A, 35B, 36C, 37C, 38B, 39E, 40B

1.8 Integers, Expressions and Order of Operations

Math Operations with Integers

Before going into order of operations, we will give an overview of adding subtracting, multiplying and dividing integers.

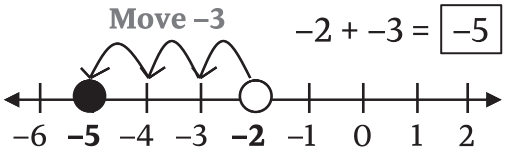
Adding Integers

Showing Addition Using Number Lines

When adding, the first number tells you where to *start* on the number line. The second number tells you how to *move or jump* on the number line.

ADD:
−2 + −3

−2 + −3
Start at −2 Move −3

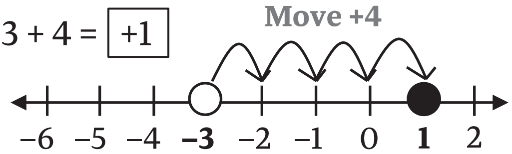


$-2 + -3 = \boxed{-5}$

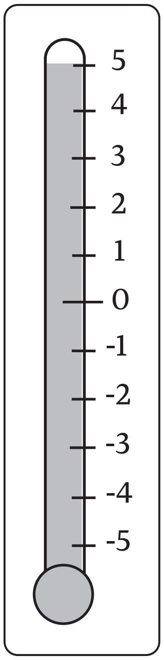
ADD:
−3 + 4

−3 + 4
Start at −3 Move +4

$-3 + 4 = \boxed{+1}$



Temperature is often used in explaining the idea of adding integers.



5 + −2 = 3

It is 5 degrees outside.
The temp. drops 2 degrees.

It will now be
3 degrees outside.

5 + −5 = 0

It is 5 degrees outside.
The temp. drops 5 degrees.

It will now be
0 degrees outside.

5 + −7 = −2

It is 5 degrees outside.
The temp. drops 7 degrees.

It is now −2 degrees.
(2 degrees below zero)

−3 + 1 = −2

It is −3 degrees outside.
The temp. rises 1 degree.

It is now −2 degrees.
(2 degrees below zero)

−3 + 8 = 5

It is −3 degrees outside.
The temp. rises 8 degrees.

It will now be
5 degrees outside.

−3 + −2 = −5

It is −3 degrees outside.
The temp. drops 2 degrees.

It is now −5 degrees.
(5 degrees below zero)

Memorization Rules for Adding Integers

If you still get confused when adding integers, there are rules you can memorize to help you add them.

Adding Integers with the Same Sign

1. Add the numbers.
2. Keep the sign.

$2 + 5$

$2 + 5 = \boxed{+7}$

$-3 + -6$

$-3 + -6 = \boxed{-9}$

Adding Integers with Different Signs

1. Ignore the signs and compare the numbers.
(Keep the sign of the bigger number.)
2. Subtract the numbers.

$-7 + 4$

Ignore the signs: $7 > 4$
The answer is negative.

$7 - 4 = 3 \Rightarrow \boxed{-3}$

Subtracting Integers

Subtraction is the same as adding by the opposite of the second number.

(To find the opposite of number, just switch its sign.)

$A - \mathbf{B}$

SAME AS

$A + \mathbf{-B}$

$1 - 7$

$1 - 7 \Rightarrow 1 + -7 = \boxed{-6}$

$8 - -2$

$8 - -2 \Rightarrow 8 + +2 = \boxed{10}$

$-3 - 5$

$-3 - 5 \Rightarrow -3 + -5 = \boxed{-8}$

The ACT also likes using temperature changes in word problems involving integers.

At the beginning of a wintry day, the temperature in Fahrenheit, was -5^{o}F.
By the end of the day, the temperature had changed to 18^{o}F.
What was the change in temperature from the beginning of the day to the end of the day?

You can probably tell that the temperature went up, so the answer will be positive.

We can set it up as an addition problem:

Begin Temp.		Rise		End Temp.
−5	+	X	=	18

The answer will be X = 23.

The change is $+23^{o}$.

You can also find the change using subtraction:

Change in Temperature = Ending Temperature − Beginning Temperature

Change = $18 - (-5) = +23^{o}$

Multiplying and Dividing Integers

When multiplying or dividing two integers:

⇨ If the signs are the same, the answer will be positive.

⇨ If the signs are different, the answer will be negative.

(+) • (+) = (+)	(+) • (−) = (−)	(+) ÷ (+) = (+)	(+) ÷ (−) = (−)
(−) • (−) = (+)	(−) • (+) = (−)	(−) ÷ (−) = (+)	(−) ÷ (+) = (−)

$-9 \bullet -8 = \boxed{72}$ $\quad -3 \bullet 10 = \boxed{-30}$ $\quad -30 \div -6 = \boxed{5}$ $\quad 28 \div -4 = \boxed{-7}$

Order of Operations

P
E
MD
AS

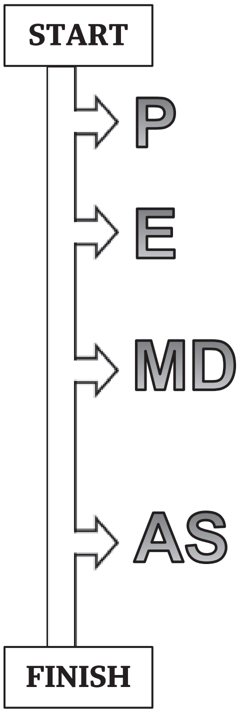
Order of operations are a set of rules that tell us the order a math expression should be worked out.

When you put numbers into your calculator, it works out the calculations using order of operations.

You can use the phrase "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" to remember the order of operations.

P
E
MD
AS

SIMPLIFY: $48 \div 4 - 5 \bullet 2^3 + (10 - 3)$



Parentheses	Work out any math problems that are inside parentheses.	$48 \div 4 - 5 \bullet 2^3 + \mathbf{(10 - 3)}$
Exponents	Work out any numbers that are raised to powers.	$48 \div 4 - 5 \bullet \mathbf{2^3} + 7$
Multiplication & **D**ivision	You work them from left to right, no matter which one comes first.	$\mathbf{48 \div 4} - 5 \bullet 8 + 7$ $12 - \mathbf{5 \bullet 8} + 7$
Addition & **S**ubtraction	You work them from *left to right*, no matter which one comes first.	$\mathbf{12 - 40} + 7$ $\mathbf{-28 + 7}$
		$\boxed{-21}$

Plugging into Expressions

When you plug numbers into an expression, always use order of operations to work it out.

What is the value of $-2x^2 + 5x$ when $x = -3$?

Plug in -3 for x:

$$\overset{-3}{\overset{\downarrow}{-2x^2}} + 5\overset{-3}{\overset{\downarrow}{x}} \Rightarrow -2(-3)^2 + 5(-3)$$

$$\begin{aligned} -2(-3)^2 + 5(-3) &= -2(9) + 5(-3) \\ &= -18 + -15 \\ &= \boxed{-33} \end{aligned}$$

If a = 2 and b = 4, what is the value of $(5a - b)^3$?

Plug in for the variables:

$$(5\overset{2}{\overset{\downarrow}{a}} - \overset{4}{\overset{\downarrow}{b}})^3 \Rightarrow (5(2) - 4)^3$$

$$\begin{aligned} (5(2) - 4)^3 &= (10 - 4)^3 \\ &= (6)^3 \\ &= \boxed{216} \end{aligned}$$

Find the value of $\frac{6}{x}$ when $x = \frac{3}{4}$

There are two ways to handle the problem.

1st Way: Change 3/4 to a decimal and plug it in for x.

2nd Way: Plug in the fraction 3/4 for x and divide.

FIRST WAY:
Change 3/4 to decimal, plug it in and divide:

$$0.75 \curvearrowright \frac{6}{x} \Rightarrow \frac{6}{0.75} = 6 \div 0.75 = \boxed{8}$$

SECOND WAY:
Plug in 3/4, change the problem to division and divide:

$$\frac{6}{\frac{3}{4}} = 6 \div \frac{3}{4} = \frac{6}{1} \bullet \frac{4}{3} = \frac{24}{3} = \boxed{8}$$

Symbolic Expressions

Symbolic expressions use symbols or pictures to tell you where numbers go in an expression.

If $(x \diamond y) = 4y + x^3$, find the value of $(5 \diamond 7)$.

$x \diamond y$
$(5 \diamond 7)$
We can see that $x = 5$ and $y = 7$.

Plug in for x and y:

$$\overset{x \quad y}{(5 \diamond 7)} \Rightarrow 4\overset{\overset{7}{\downarrow}}{y} + \overset{\overset{5}{\downarrow}}{x}{}^3 \Rightarrow 4(7) + (5)^3$$

Simplify the expression:

$$4(7) + (5)^3 = 28 + 125 = \boxed{153}$$

If $(g \$ h) = g - 2^h$, find the value of $9 \$ (6 \$ 0)$.

Follow order of operations by first finding the answer to the expression in parentheses:

$$\overset{g \$ h}{(6 \$ 0)} \Rightarrow \overset{\overset{6}{\downarrow}}{g} - 2^{\overset{\overset{0}{\downarrow}}{h}} \Rightarrow 6 - 2^0 = 6 - 1 = 5$$

Now plug 5 into the expression to finish the problem:

$$9 \$ \overset{5}{\overbrace{(6 \$ 0)}} \Rightarrow \overset{g \$ h}{9 \$ 5} \Rightarrow \overset{\overset{9}{\downarrow}}{g} - 2^{\overset{\overset{5}{\downarrow}}{h}} \Rightarrow 9 - 2^5 = 9 - 32 = \boxed{-23}$$

Variable Expressions and Like Terms

In general, math expressions can contain variables, numbers and math symbols.

Math expressions have **terms**. Terms are separated by (+) and (–) signs.

This expression has five terms:
$-9x + 7 - x - 6 + 8x$

$$\underset{\substack{\downarrow \\ \text{Term}}}{\mathbf{-9x}}\ \underset{\substack{\downarrow \\ \\ \text{Term}}}{+7}\ \underset{\substack{\downarrow \\ \text{Term}}}{\mathbf{-x}}\ \underset{\substack{\downarrow \\ \\ \text{Term}}}{-6}\ \underset{\substack{\downarrow \\ \text{Term}}}{\mathbf{+8x}}$$

The terms are:
$-9x, 7, -x, -6, 8x$

You can move terms around as long as you do not change the signs of the terms.

The signs of the terms don't change as we move them around:

$-9x + 7 - x - 6 + 8x \Rightarrow$

$8x - x - 9x + 7 - 6$

$-9x - 6 + 8x - x + 7$

$7 - x - 6 + 8x - 9x$

Like Terms are terms that we can combine using addition or subtraction.

Terms with the same variable can be combined:	Numbers can be combined with other numbers:	The examples below are NOT like terms:
$6x - 2x \Rightarrow 4x$	$9 - 11 \Rightarrow -2$	$4x - 5y$ (Can't be combined)
$2.5y + y \Rightarrow 3.5y$	$10 + 8.7 \Rightarrow 18.7$	$3x + 3$ (Can't be combined)

The Distributive Property

When a number is multiplied by a group in parentheses that is added or subtracted, you can "distribute" the number into the group.

$9(x + 5) \Rightarrow 9(x + 5) \Rightarrow \boxed{9x + 45}$

$9 \bullet x = 9x \qquad 9 \bullet 5 = +45$

$7(a - 6) \Rightarrow 7(a - 6) \Rightarrow \boxed{7a - 42}$

$7 \bullet a = 7a \qquad 7 \bullet -6 = -42$

$-3(b + 2) \Rightarrow -3(b + 2) \Rightarrow \boxed{-3b - 6}$

$-3 \bullet b = -3b \qquad -3 \bullet 2 = -6$

$-(c - 8) \Rightarrow -1(c - 8) \Rightarrow \boxed{-c + 8}$

$-1 \bullet c = -1c \qquad -1 \bullet -8 = +8$

Combining Like Terms

Be careful with negatives when combining like terms.

SIMPLIFY:
$7x - 5 - 2y + 9 - y + 3x$

Combine like terms:

$7x + 3x \Rightarrow 10x$

$-2y - 1y \Rightarrow -3y$

$-5 + 9 \Rightarrow 4$

$\Rightarrow \boxed{10x - 3y + 4}$

SIMPLIFY:
$7(2x + 3) - 4(9 - x)$

Distribute:

$7(2x + 3) - 4(9 - x)$

$14x + 21 - 36 + 4x$

Combine like terms:

$14x + 4x \Rightarrow 18x$

$21 - 36 \Rightarrow -15$

$\boxed{18x - 15}$

Practice Problems

1. Simplify the expression by combining like terms:
$-5(3x - 2) - (8 - 11x)$

A. $-4x - 18$
B. $-4x + 2$
C. $-4x + 18$
D. $-26x - 18$
E. $-26x + 2$

2. What is the value of the expression below?
$(1 + 36 \div 18 \times 2)^2 - 2^3 + 1$

A. 5
B. 9
C. 16
D. 18
E. 20

3. If $a = -2$ and $d = 3$, the simplified form of the expression $ab + cd + bd + ac$ is:

A. b + c
B. b – c
C. b – 5c
D. 5b + 5c
E. –5b – 5c

4. What is the answer to the expression below?
$1{,}000 \div 2 \div 4 \times 3 \times 2 - 100 - 50 + 25 + (-1)^9$

A. –43
B. –42
C. 120
D. 623
E. 624

5. Let x be the largest integer that is less than –3.
Let y be the smallest integer that is greater than –6.
What is the value of $x - y$?

A. –3
B. –1
C. 1
D. 3
E. 5

6. Let x be an integer in the range $-7 \leq x \leq 6$.
Let y be an integer in the range $-6 \leq y \leq 7$.
What is the largest possible value of $\frac{x}{y}$?

A. 4
B. 5
C. 6
D. 7
E. 8

7. Simplify the expression by combining like terms:
$4(x + 6) - 3(x + 7)$

A. $x - 3$
B. $x + 3$
C. $x + 13$
D. $7x + 3$
E. $7x + 13$

8. Use order of operations to simplify the expression:

$$\boxed{\frac{7 + 3 \times -5 - 8 \div 4}{10 \div 2 + 3 \times 9 - 7(2)}}$$

A. –2/3
B. –5/9
C. 1/9
D. 2/3
E. 5/9

9. Simplify the expression by combining like terms:
$(5c - d + 8e) - (c - d - 9e)$

A. $4c - 2d - e$
B. $4c - 2d + e$
C. $4c - 2d + 17e$
D. $4c - e$
E. $4c + 17e$

10. Which of the following expressions have a positive value for all x and y such that $x > 0$ and $y < 0$?

A. $y - x$
B. $x + y$
C. x^3y
D. $\frac{x^2}{y}$
E. $\frac{x}{y^2}$

11. Find the value of the expression $A(B - C)^D$ when A = –4, B = 2, C = –3 and D = 3.

A. –500
B. –60
C. –12
D. –4
E. 4

12. The symbol @ is used to define the expression $(x @ y) = x^2 - xy$. Which answer choice below is *not* equivalent to (10 @ 4)?

A. (12 @ 7)
B. (15 @ 11)
C. (16 @ 12)
D. (20 @ 17)
E. (30 @ 28)

13. Use order of operations to simplify the expression:

$$\boxed{\frac{8 - 2^2}{(8 - 2)^2} + \frac{4 - 11 \div 11 + 4}{4 \div (11 \div 11) \div 4}}$$

A. 1/3
B. 4/5
C. 11/37
D. 64/9
E. 8

14. From 3:40pm to 4:30pm, the temperature in a city changed from 12^{o}F to -8^{o}F. What was the average drop in temperature in degrees Fahrenheit per minute?

A. –2.5
B. –0.8
C. –0.6
D. –0.5
E. –0.4

15. The symbol & is used to define the expression $(A \& B) = \dfrac{-A}{B - A}$. Find the value of (–9 & 3).

A. –3
B. –3/2
C. 3/4
D. 3/2
E. 3

16. Evaluate $x^4 + x^3 - x^2 - x - 6$ when $x = -2$.

A. –4
B. 0
C. 8
D. 12
E. 16

17. Two cities are observed based on random temperature readings in the morning and in the evening. The data from City A and City B are shown in the table. All temperatures are expressed in degrees Fahrenheit. Which of the statements are true?

City	Morning Temperature	Evening Temperature
A	9^{o}F	-17^{o}F
B	-4^{o}F	21^{o}F

I. The change in temperature from the morning to the evening of City A is -26^{o}F.
II. The change in temperature from the morning to the evening of City B is $+25^{o}$F.
III. The morning temperature of City A is 13^{o}F larger than the morning temperature of City B.

A. I only
B. I and II only
C. I and III only
D. II and III only
E. I, II and III

18. If $x = 2$ and $y = 4$, what is the value of $(6x + y)^2$?

A. 64
B. 144
C. 256
D. 576
E. 676

19. Let the symbol Δ be part of an operation such that $(A \, \Delta \, B) = 3^{A+B} + 1$. Find the value of (–4 Δ 6) Δ –10.

A. 2
B. 5
C. 8
D. 11
E. 14

20. Find the value of $(x^3 - \sqrt{y})^2 + (16 - 2y)^3$ when $x = 3$ and $y = 9$.

A. 28
B. 232
C. 233
D. 568
E. 772

21. Evaluate a + b − c − d when a = –1, b = –2, c = –3 and d = –4.

A. –10
B. –4
C. –2
D. 4
E. 9

22. What is the value of $4x^3 - 10x$ when $x = \frac{1}{2}$?

A. –9/2
B. –9/4
C. –3
D. 9/2
E. 3

23. The formula $C = \frac{5}{9}(F + 32)$ can be used to convert a temperature in degrees Fahrenheit (^oF) to degrees Celsius (^oC). The formula K = C + 273 can be used to convert a temperature from degrees Celsius to Kelvin (K). What is the temperature in Kelvin if the temperature outside is -50^oF?

A. 263
B. 283
C. 291
D. 309
E. 318

24. Evaluate $2 - (x - y)(x + y)$ when $x = -7$ and $y = 4$.

A. –119
B. –31
C. –7
D. 11
E. 35

25. Distribute and simplify the expression:
$\frac{1}{4}(8x - 4y + 16z) + \frac{2}{3}(9y + 6x - 3z)$

A. $6x + 2y - 6z$
B. $6x + 2y + 2z$
C. $6x + 5y + 2z$
D. $6x + 6y + 2z$
E. $6x + 6y - 6z$

26. What is the result of $-x^2 - x + 6$ when $x = 2$?

A. 0
B. 4
C. 6
D. 8
E. 12

27. Given that $x = -7$, $y = -8$ and $z = 3/5$, find the result after plugging into the expression $\frac{x + y}{z}$.

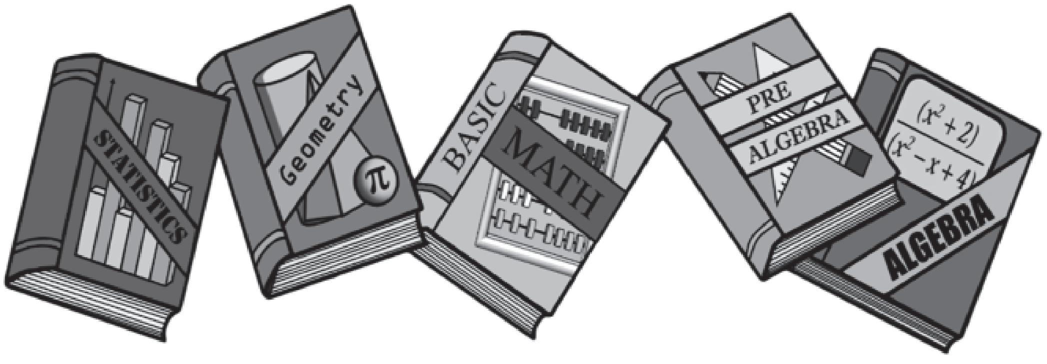
A. –25
B. –5/3
C. 1/3
D. 5/3
E. 25

Solutions: 1B, 2D, 3A, 4E, 5C, 6D, 7B, 8B, 9E, 10E, 11A, 12C, 13D, 14E, 15C, 16B, 17E, 18C, 19A, 20D, 21D, 22A, 23A, 24B, 25C, 26A, 27A

Math on the Fly!

CHAPTER 3

Plane and Coordinate Geometry



3.13 Slopes of Lines

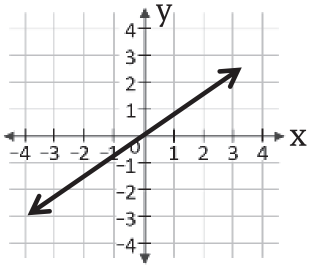
Overview of Slope

The “slant” or “steepness” of a line is called the **slope**.

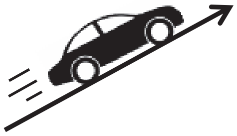
When looking at the slope of a line, you view the line from left to right.

There are 4 basic types of slopes that a line can have:

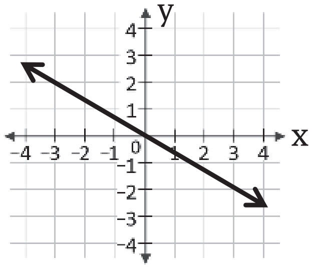
Positive Slope



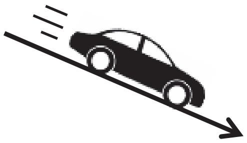
Driving or moving upwards



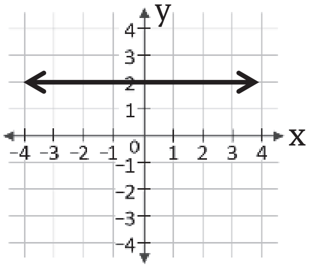
Negative Slope



Driving or moving downwards

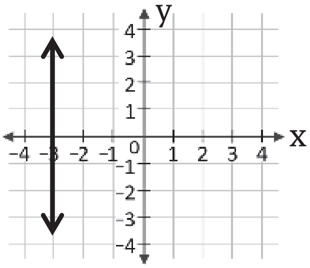


Zero Slope

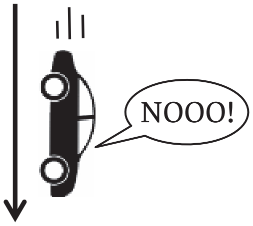


Driving on a flat road

Undefined Slope



We can’t drive on the road at all!



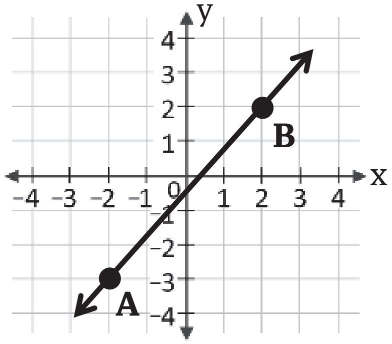
Using “Rise over Run” to Find Slope

We can find the exact slope of a line by counting the RISE and RUN of the line.

$$m = \frac{\text{RISE} \updownarrow}{\text{RUN} \leftrightarrow}$$

The steps you move up or down is called the RISE.

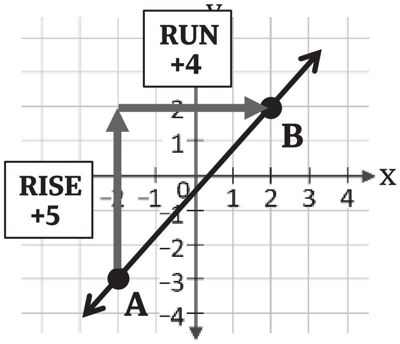
The steps you move left or right is called the RUN.



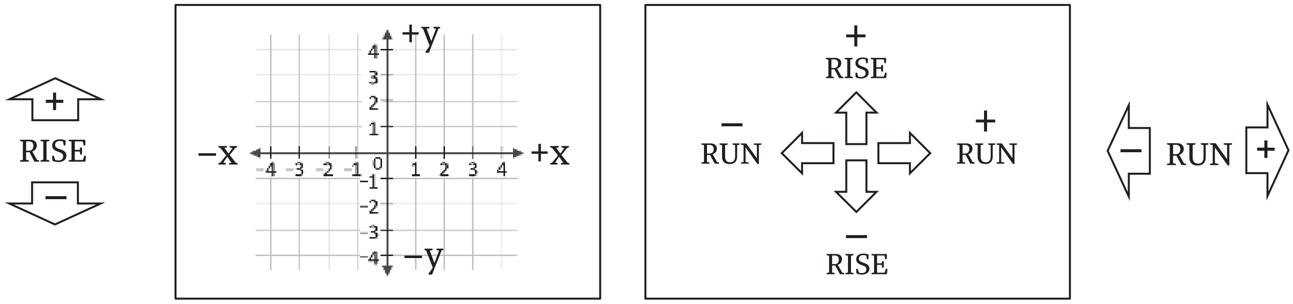
Slope of the Line:

$$m = \frac{5}{4}$$

To go from point A to B,
RISE +5 steps
and RUN +4 steps

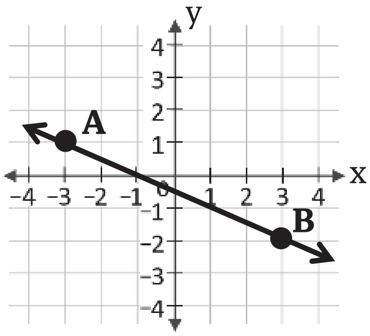


You need to be mindful of the signs when you do your rise and run.
The signs of the rise and run are no different than the signs of the x–axis and y–axis.



When you find slope on a graph,
we recommend you do your rise first, and then do the run.

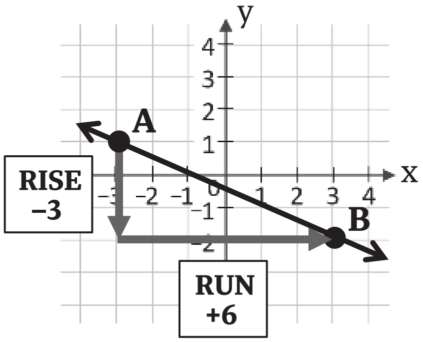
Find the slope of the line:



The slope of the line is –1/2.

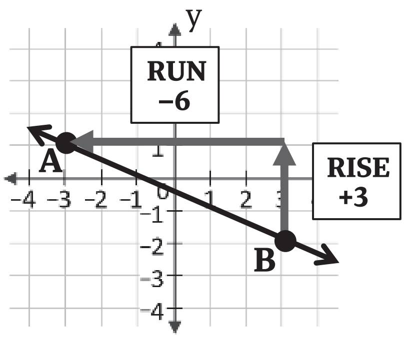
We will find the slope two different ways.

Move from A to B



$$m = \frac{\text{RISE}}{\text{RUN}} = \frac{-3}{+6} = -\frac{1}{2}$$

Move from B to A



$$m = \frac{\text{RISE}}{\text{RUN}} = \frac{+3}{-6} = -\frac{1}{2}$$

The Slope Formula

Another way to find the slope between two points is to use the slope formula.

First Point:	Second Point:
(x_1, y_1)	(x_2, y_2)
1st X 1st Y	2nd X 2nd Y

$$m = \frac{y_1 - y_2}{x_1 - x_2} \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{\text{Change in y's}}{\text{Change in x's}}$$

Slope is sometimes described as the “rate of change” between the y–values and x–values.

It does not matter which formula you use.
Just make sure you subtract the y’s and x’s in the *same order*.

Find the slope between $(\overset{x_1}{-2},\overset{y_1}{1})$ and $(\overset{x_2}{2},\overset{y_2}{4})$

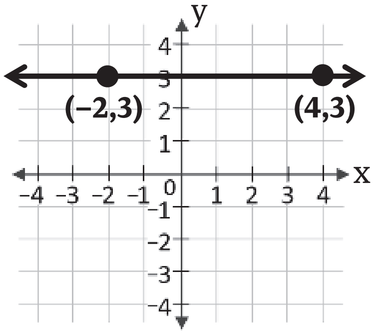
$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1-4}{-2-2} = \frac{-3}{-4} \Rightarrow \boxed{\frac{3}{4}}$$

Find the slope between $(\overset{x_1}{0},\overset{y_1}{6})$ and $(\overset{x_2}{2},\overset{y_2}{-4})$

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{6-(-4)}{0-2} = \frac{10}{-2} \Rightarrow \boxed{-5}$$

Find the slope between $(\overset{x_1}{4},\overset{y_1}{3})$ and $(\overset{x_2}{-2},\overset{y_2}{3})$

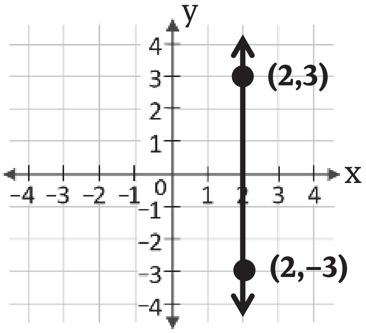
$$m = \frac{3-3}{4-(-2)} = \frac{0}{6} = \boxed{0}$$



The slope is 0.
The line is horizontal.

Find the slope between $(\overset{x_1}{2},\overset{y_1}{3})$ and $(\overset{x_2}{2},\overset{y_2}{-3})$

$$m = \frac{3-(-3)}{2-2} = \frac{6}{0} = \boxed{\text{undefined}}$$



The slope is undefined.
The line is vertical.

Division with Zero

Students mess up division with zero all the time.

It is OK to have zero on top.

$$\frac{0}{\text{Number}} = \boxed{0}$$

It is NOT OK to have zero on the bottom.

$$\frac{\text{Number}}{0} = \boxed{\text{Undefined}}$$

(Use a calculator to prove it to yourself that these statements are true.)

Why is a fraction undefined if it has zero in the denominator?
Below is the easiest way to think about it:

$$\frac{12}{3} = 4$$

12 ÷ **3** = **4**
is the same as saying
3 × **4** = 12

$$\frac{45}{5} = 9$$

45 ÷ **5** = **9**
is the same as saying
5 × **9** = 45

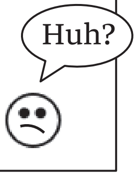
$$\frac{0}{6} = 0$$

0 ÷ **6** = **0**
is the same as saying
6 × **0** = 0

Now look at the problem below:

$$\frac{6}{0} = ?$$

6 ÷ **0** = **?**
is the same as saying
0 × **?** = 6

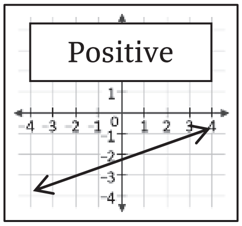


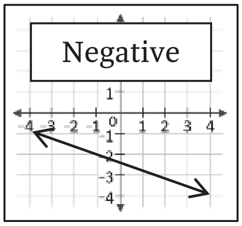
There is no number you can multiply with 0 that will give you an answer of 6.

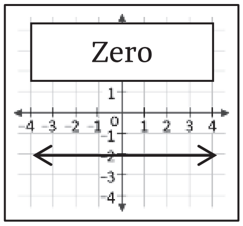
$\frac{6}{0}$ is undefined

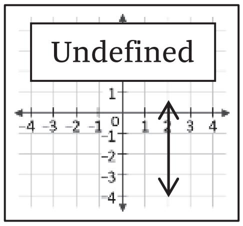
Review of Slope

Slope tells you the steepness of a line.
The slope of a line can be positive, negative, zero or undefined.

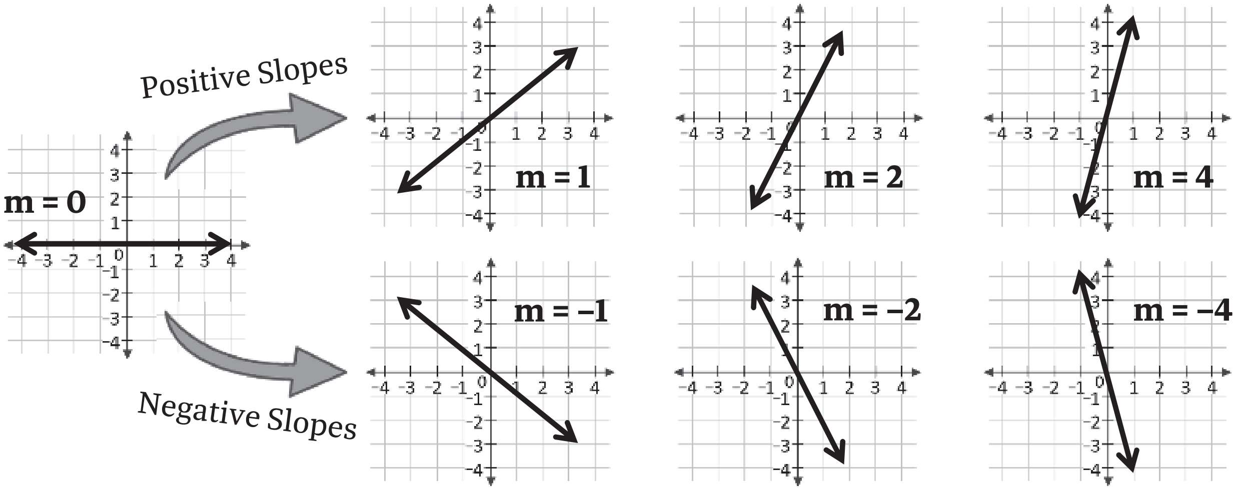








As the slopes become more positive or become more negative, the lines get steeper.

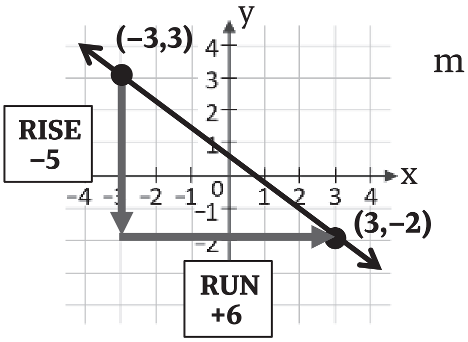


Make sure you know how to find the slope between 2 points
using a graph and by using the formula.

$$m = \frac{\text{Change in y's}}{\text{Change in x's}} = \boxed{\frac{\text{RISE} \updownarrow}{\text{RUN} \leftrightarrow}} = \boxed{\frac{y_1 - y_2}{x_1 - x_2} \text{ or } \frac{y_2 - y_1}{x_2 - x_1}}$$

Find the slope between (–3,3) and (3,–2)

Finding Slope with a Graph



$$m = \frac{\text{RISE}}{\text{RUN}} = \frac{-5}{+6} = \boxed{-\frac{5}{6}}$$

Finding Slope with the Formula

$$\underset{(-3,3)}{\overset{x_1\ y_1}{\downarrow\ \downarrow}} \qquad \underset{(3,-2)}{\overset{x_2\ y_2}{\downarrow\ \downarrow}}$$

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{3 - (-2)}{-3 - 3} = \frac{5}{-6} \Rightarrow \boxed{-\frac{5}{6}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 3}{3 - (-3)} = \frac{-5}{6} \Rightarrow \boxed{-\frac{5}{6}}$$

Practice Problems

1. Below is information describing lines L_1, L_2 and L_3:

Line L_1: Passes through the points (–3,–1) and (3,–4)
Line L_2: Passes through the points (6,–1) and (3,5)
Line L_3: Passes through the points (4,10) and (–2,7)

Which of the statements is a true statement?

A. Lines L_1 and L_3 have the same slope.
B. The slopes of all three lines are negative.
C. The slopes of all three lines are the same.
D. The slopes of all three lines are different.
E. The slopes of all three lines are integers.

2. A line that passes through the points (6,7) and (9,y) has the same slope as a line that passes through the points (5,–8) and (3,0). What is the value of y?

A. –19
B. –5
C. 25/4
D. 5
E. 19

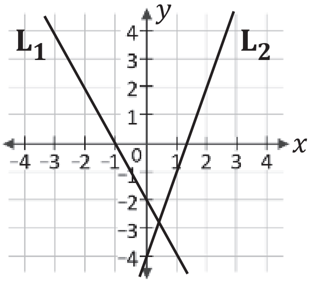
3. Line P passes through the points (–2,–8) and (2,–8). Line Q passes through the points (–4,7) and (–4,–7). Which statements are true about the lines?

I. Line P has a slope that is undefined.
II. Line Q has a slope that is undefined.
III. One of the lines has a slope of zero.

A. I only
B. II only
C. I and III only
D. II and III only
E. I, II and III

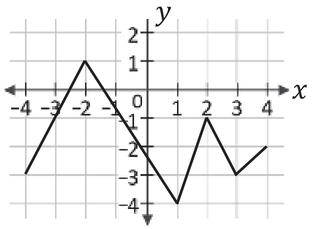
4. Lines L_1 and L_2 are shown below. Let m_1 be the slope of line L_1. Let m_2 be the slope of line L_2 . What is the value of $m_1 - m_2$?

A. –5
B. –7/3
C. –1
D. 1
E. 5



5. Below is a graph of a curve from $x = -4$ to $x = 4$. Between what values of x is the rate of change at its greatest value?

A. From $x = -4$ to $x = -2$
B. From $x = -2$ to $x = 1$
C. From $x = 1$ to $x = 2$
D. From $x = 2$ to $x = 3$
E. From $x = 3$ to $x = 4$



6. The ratio of the rise and run of a line is $-4/5$. If the line passes through (–4,2), another point on the line must be:

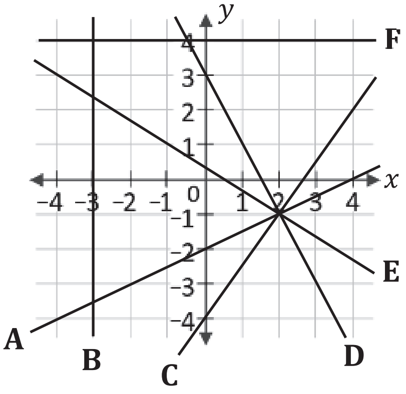
A. (–8, –3)
B. (1, –2)
C. (–8, 7)
D. (0,7)
E. (–8, 3)

7. Let x be an integer. What should the value of x be so that the line that passes through the points (x,5) and (–8,–4) has a slope of 3/4?

A. –20
B. –4
C. 4
D. 8
E. 20

8. In the diagram are 6 distinct lines. Each line has been assigned a letter from A to F to aid in distinguishing them from each other. If one of the lines is chosen at random, what is the probability that the slope of the line will be between –1 and 1?

A. 1/6
B. 2/6
C. 3/6
D. 4/6
E. 5/6



Solutions: 1D, 2B, 3D, 4A, 5C, 6B, 7C, 8C